# Continuation Anatomy of Periodic Navier–Stokes Breakdown

A machine-checked $H^2$ theory of finite-energy regularity escape

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**Abstract**

We develop a machine-checked continuation theory for smooth forced periodic three-dimensional Navier–Stokes data and use it to characterize the structure of any periodic classical breakdown datum. The central result is independent of a particular singular construction. Whenever smooth physical periodic data admit no global smooth classical solution, the associated native $H^2$ evolution has finite maximal lifespan and

$$\|u(t)\|_{H^2} \longrightarrow \infty \qquad (t \uparrow T_{\max}).$$

At the same time, the kinetic energy

$$E(t) = \frac{1}{2}\|u(t)\|_{L^2}^2$$

remains uniformly finite:

$$\sup_{t<T_{\max}} E(t) < \infty.$$

For every finite Fourier cutoff $K$, the $H^2$ mass above $K$ tends to infinity, while for every fixed heat scale $a > 0$, the regularized state $e^{a\Delta}u(t)$ remains uniformly $H^2$-controlled. Finite maximality also excludes every uniform positive bounded-control continuation certificate near the endpoint. The resulting theorem package identifies periodic strong breakdown as fine-scale regularity escape and collapse of strong continuation capacity inside a finite-energy flow.

The proof proceeds through an exact classical/native bridge. Smooth real periodic initial data descend to Fourier/Sobolev carriers, preserve incompressibility and reality, inhabit every finite Sobolev order, and return exactly to the original classical data after reconstruction. Smooth periodic forcing admits the corresponding exact round trip, including the $2\pi$ derivative normalization, viscosity conversion, and force rescaling. Consequently a hypothetical infinite native trajectory would produce a global classical solution for the original datum itself.

A recently announced OpenAI construction supplies a first concrete application: OpenAI reports a formally verified proof of the forced breakdown alternatives (C) and (D) in the official Clay formulation. If accepted, either alternative resolves the mathematical problem as posed, while the historically central zero-force regularity question remains logically distinct. The continuation anatomy proved here stands independently: it states what any periodic smooth classical breakdown datum satisfying the declared interface must look like in the native $H^2$ theory.

# 1 Introduction

A breakdown theorem and a continuation theorem answer different parts of the same mathematical question. A breakdown theorem supplies data for which global smooth classical evolution fails. A continuation theorem identifies the analytic structure that makes extension impossible. The present paper develops the second object in a form precise enough to receive the first.

The setting is the forced incompressible Navier–Stokes system on the unit three-torus,

$$\partial_t u + (u \cdot \nabla)u = \nu \Delta u - \nabla p + f, \qquad \nabla \cdot u = 0, \qquad \nu > 0. \tag{1}$$

The native state space is a physical periodic $H^2$ carrier. Leray projection, physical quadratic multiplication, tensor divergence, heat evolution, local Duhamel dynamics, restart, overlap uniqueness, gluing, maximal lifespan, all-orders bootstrap, pressure recovery, and return to the classical periodic equation are represented by explicit Lean theorems.

Let $\mathsf{BreakdownData}(\nu)$ denote smooth real one-periodic initial velocity and smooth one-periodic forcing satisfying the comparator hypotheses, together with the assertion that no global smooth periodic solution exists for those same data. The exact classical/native bridge and the maximal continuation theory yield the implication surface

$$\boxed{\begin{array}{c}
\text{smooth periodic classical breakdown datum} \\
\Downarrow \\
\text{exact classical/native round trip} \\
\Downarrow \\
T_{\max} < \infty \\
\Downarrow \\
\|u(t)\|_{H^2} \to \infty \\
\Downarrow \\
\sup\limits_{t<T_{\max}} E(t) < \infty \quad \mathsf{H}_{2,>K}(u(t)) \to \infty \ \ \forall K \quad \sup\limits_{t<T_{\max}} \mathsf{H}_2(e^{a\Delta}u(t)) < \infty \ \ \forall a > 0 \\
\text{where } E(t) = \dfrac{1}{2}\|u(t)\|_{L^2}^2 \\
\Downarrow \\
\text{uniform strong-continuation capacity disappears at the endpoint.}
\end{array}} \tag{1.2}$$

This diagram is the center of the paper. Section 2 proves the finite-breakdown mechanism. Sections 3 and 4 build the native $H^2$ evolution and the exact classical/native round trip required to apply it to external data. Section 5 then proves the single bridge-dependent implication

$$\neg \mathsf{GlobalSmoothPeriodic} \Longrightarrow T_{\max} < \infty,$$

after which the mechanism in Section 2 applies automatically.

The exact round trip carries much of the mathematical weight. Smooth classical periodic fields descend to Fourier/Sobolev carriers, preserve reality and incompressibility, inhabit every finite

Sobolev order, and return to the same physical field after reconstruction. The unit-torus derivative convention contributes the factor $2\pi$, so time, viscosity, and forcing are rescaled exactly. The native strong trajectory then bootstraps to smooth physical representatives and periodic pressure. A hypothetical infinite native trajectory therefore produces a global classical solution for the original comparator datum itself.

The maximal theory also gives a precise meaning to breakdown. Throughout the maximal interval, Eq. (1) remains the governing law and the trajectory remains a strong solution. The endpoint is the maximal horizon of the declared strong $H^2$ class: every bounded control sufficient for a uniform restart disappears as that horizon is approached. The regularity class reaches its maximal horizon.

A newly announced OpenAI result supplies a timely application [5, 6, 7]. OpenAI reports a proof of the forced breakdown alternatives (C) and (D) in Fefferman's official Clay formulation. The present paper uses that announcement only as an application of the general theorem package; the continuation anatomy itself depends solely on the periodic breakdown premise. The current Clay status and the logical separation between the forced and zero-force statements are treated in Sections 6 and 7.

### 1.1 Relation to nearby work

The weak existence theory begins with Leray's construction of finite-energy weak solutions [8]. Partial regularity results of Caffarelli, Kohn and Nirenberg constrain the possible singular set of suitable weak solutions [9]. Critical continuation criteria, including the $L^\infty_t L^3_x$ result of Escauriaza, Seregin and Šverák, identify stronger conditions under which regularity persists [10].

Several developments illuminate neighboring failure regimes. Tao constructed finite-time blow-up for an averaged three-dimensional Navier–Stokes equation which retains the global energy cancellation of the original nonlinearity [11]. Buckmaster and Vicol established nonuniqueness in weak finite-energy classes [12]. Albritton, Brué and Colombo produced nonunique forced Leray solutions with zero initial velocity under a forcing regime singular at the initial time [13]. The announced OpenAI construction occupies a different lane: the force is smooth, the flow begins from a smooth state, and the claimed singularity occurs inside the classical forced equation [6].

The present development addresses the implication layer surrounding any smooth periodic breakdown datum. Its formal novelty lies at the analytic interfaces which paper proofs often compress into an identification: physical multiplication and Fourier convolution, the domain of the Laplacian, heat smoothing with the zero mode retained correctly, mild and strong evolution, maximal-history assembly, the $2\pi$ classical/Fourier normalization, pressure recovery, and the exact external-data round trip. The proof assistant forces each bridge to carry its own typed authority.

## 2 The main continuation theorem package

We first state the theorem package independently of the OpenAI construction.

Let an $H^2$ evolution scenario consist of positive viscosity and forcing with the regularity required by the local strong theory. Let the initial datum be physical: $H^2$, real, and divergence-free. The local theory gives a unique maximal strong trajectory

$$u : [0, T_{\max}) \to H^2.$$

### 2.1 Maximal lifespan and eventual $H^2$ divergence

Local existence is uniform under bounded state and forcing control. More precisely, if

$$\|u(t_0)\|_{H^2} \le M \tag{2}$$

and the forcing is bounded by $F$ on the prescribed unit future window, then there exists

$$\delta = \delta(\nu, M, F) > 0 \tag{3}$$

such that the strong evolution can be restarted from $u(t_0)$ for at least $\delta$.

Suppose $T_{\max} < \infty$. A uniform $H^2$ bound near $T_{\max}$, together with smooth forcing on the finite time window, supplies fixed $M$ and $F$. Equation (3) then produces one positive restart margin valid arbitrarily close to the endpoint. Choosing $t_0$ with

$$T_{\max} - t_0 < \delta$$

extends the solution beyond maximality. Hence

$$T_{\max} < \infty \implies \forall M\ \exists t_M < T_{\max}\ \forall t \in (t_M, T_{\max}) : M < \|u(t)\|_{H^2}. \tag{4}$$

Equivalently,

$$T_{\max} < \infty \implies \|u(t)\|_{H^2} \longrightarrow \infty. \tag{5}$$

[1]

The filter statement in Eq. (5) gives eventual divergence. Every finite threshold is eventually exceeded and remains exceeded near the endpoint.

### 2.2 Finite kinetic energy persists

Define

$$\mathcal{E}(u) = \frac{1}{2}\sum_{i=1}^{3} \|u_i\|_{L^2}^2. \tag{6}$$

For physical strong trajectories, the quadratic term cancels in the energy pairing and the normalized Laplacian produces the dissipation identity

$$\frac{d}{dt}\mathcal{E}(u(t)) + \nu\|\nabla u(t)\|_{L^2}^2 = \mathrm{Re}\langle f(t), u(t)\rangle. \tag{7}$$

[2]

[1] *Formal counterpart:* `Tier30.Research.NavierStokes.Evolution.finiteMaximalLifespan_H2Norm_tendsto_atTop`.

[2] *Formal counterpart:* `Tier30.Research.NavierStokes.FailureMechanism.quadraticResponse_energyPairing_zero`.

Smooth forcing is bounded on every compact time interval. Cauchy–Young control and Grönwall yield

$$T_{\max} < \infty \quad \Longrightarrow \quad \exists E < \infty \, \forall t < T_{\max} : \|u(t)\|_{L^2} \le E. \tag{8}$$

[3]

Thus finite breakdown admits the simultaneous statement

$$\boxed{\sup_{t<T_{\max}} \|u(t)\|_{L^2} < \infty, \qquad \|u(t)\|_{H^2} \to \infty.} \tag{9}$$

Finite kinetic energy remains available while derivative-weighted regularity escapes.

### 2.3 High-frequency $H^2$ escape

Let

$$\mathsf{H}_2(u) = \sum_{i=1}^{3} \sum_{k\in\mathbb{Z}^3} (1+|k|^2)^2 |\widehat{u}_i(k)|^2. \tag{10}$$

For $K \ge 0$, split

$$\mathsf{H}_2(u) = \mathsf{H}_{2,\le K}(u) + \mathsf{H}_{2,>K}(u). \tag{11}$$

The low-frequency part obeys

$$\mathsf{H}_{2,\le K}(u) \le (1+K^2)^2 \mathsf{L}_2(u), \tag{12}$$

where $\mathsf{L}_2$ is the summed Fourier $L^2$ mass.

Equation (8) controls Eq. (12) uniformly for each fixed $K$. Equation (5) drives the total $H^2$ mass to infinity. Therefore

$$\boxed{\forall K < \infty, \qquad \mathsf{H}_{2,>K}(u(t)) \longrightarrow \infty.} \tag{13}$$

[4]

Every fixed finite Fourier window remains bounded in the regularity budget supplied by finite energy. The divergent $H^2$ mass cannot remain confined below any fixed spectral cutoff.

### 2.4 Every fixed positive heat scale remains regular

Let

$$F_a = e^{a\Delta}, \qquad a > 0.$$

The native Fourier multiplier estimate gives

$$\mathsf{H}_2(F_a u) \le (1+a^{-1})^2 \, \mathsf{L}_2(u). \tag{14}$$

[5]

[3] *Formal counterpart:* `Tier30.Research.NavierStokes.FailureMechanism.finiteMaximalLifespan_L2Norm_bounded`.

[4] *Formal counterpart:* `Tier30.Research.NavierStokes.FailureMechanism.finiteMaximalLifespan_highFrequencyH2_tendsto_atTop`.

[5] *Formal counterpart:* `Tier30.Research.NavierStokes.FailureMechanism.heat_H2_le_L2_fixedPositiveScale`.

Combining Eqs. (8) and (14),

$$\boxed{\forall a > 0, \qquad \sup_{t<T_{\max}} \mathsf{H}_2(F_a u(t)) < \infty.} \tag{15}$$

[6]

This is an analytic statement about heat-regularized presentations of the trajectory. It makes no identification with a physical measurement apparatus and assigns no independent filtered Navier–Stokes dynamics to the regularized fields.

The scale constant itself records the limiting geometry:

$$(1 + a^{-1})^2 \longrightarrow \infty \qquad (a \downarrow 0). \tag{16}$$

Finite maximality therefore excludes a uniform near-endpoint $H^2$ bound across all sufficiently small positive heat scales.[7]

Each positive scale carries regularity. Uniform passage to the source scale carries the singular obstruction.

### 2.5 Continuation capacity collapses

The uniform local existence theorem selects a positive certified lifespan from any state bound $M$ and force bound $F$:

$$\delta_{\text{cert}}(\nu, M, F) > 0. \tag{17}$$

For every submaximal state, an available force bound therefore supplies a positive local continuation certificate.[8]

At finite maximality, the actual remaining geometric lifetime tends to zero. More structurally, finite maximality excludes fixed bounds $M, F$ and a fixed positive reserve $\delta$ which remain valid near the endpoint. Such a certificate would restore a uniform restart interval and force global continuation. Hence

$$\boxed{\text{every uniform positive bounded-control continuation certificate disappears near } T_{\max}.} \tag{18}$$

[9]

The canonical structure

`FiniteBreakdownMechanism`

packages finite lifespan, bounded energy, eventual $H^2$ divergence, high-frequency escape, fixed positive-scale regularity, failure of uniform fine-scale control, and collapse of uniform continuation reserve in one theorem-owned object.[10]

[6] *Formal counterpart:* `Tier30.Research.NavierStokes.FailureMechanism.finiteMaximalLifespan_fixedHeatScale_H2_bounded`.

[7] *Formal counterpart:* `Tier30.Research.NavierStokes.FailureMechanism.finiteMaximalLifespan_not_uniformFineScaleH2Control`.

[8] *Formal counterpart:* `Tier30.Research.NavierStokes.FailureMechanism.maximalPath_positiveContinuationReserve`.

[9] *Formal counterpart:* `Tier30.Research.NavierStokes.FailureMechanism.finiteMaximalLifespan_continuationReserve_collapses`.

[10] *Formal counterpart:* `Tier30.Research.NavierStokes.FailureMechanism.finiteMaximalLifespan_has_failureMechanism`.

# 3 Native periodic $H^2$ evolution

The theorem package above rests on an end-to-end periodic strong-solution theory.

## 3.1 The physical Sobolev carrier

The velocity field is represented by Fourier coefficients

$$\widehat{u}_i(k), \qquad k \in \mathbb{Z}^3, \qquad i \in \{1,2,3\}.$$

The $H^2$ condition is

$$\sum_{k\in\mathbb{Z}^3} (1+|k|^2)^2 |\widehat{u}_i(k)|^2 < \infty \qquad \text{for each } i. \tag{19}$$

Reality becomes conjugate symmetry,

$$\widehat{u}(-k) = \overline{\widehat{u}(k)}, \tag{20}$$

and incompressibility becomes

$$k \cdot \widehat{u}(k) = 0. \tag{21}$$

The periodic Leray projector is constructed modewise as orthogonal projection onto the divergence-free fiber. It is bounded on the native $L^2$ carrier, idempotent, fixes divergence-free states, preserves the finite Sobolev orders used in the development, and commutes with the heat presentation.

The quadratic response begins from the physical torus product. Its Fourier coefficient is then proved equal to convolution. This ordering makes the nonlinear response an actual physical Sobolev product with a theorem-owned Fourier representation.

## 3.2 Parabolic local evolution

The local theory follows the semigroup geometry of the equation. The nonlinear response loses one derivative. Positive-time heat evolution returns that derivative with the integrable singularity

$$\|e^{(t-s)\Delta} g\|_{H^2} \lesssim (t-s)^{-1/2} \|g\|_{H^1}. \tag{22}$$

The Duhamel map therefore contracts on a sufficiently short interval in $C_t H^2_x$.

The resulting mild solution is identified with the $L^2$-differentiable strong law. Reality and incompressibility persist, uniqueness holds on overlapping intervals, and quantitative continuous dependence produces local Hadamard well-posedness.[11]

One formal correction is worth recording. The inhomogeneous heat carrier contains an undamped zero Fourier mode. The smoothing theorem used by the evolution proof therefore retains the zero-mode contribution or works on the mean-zero derivative response. The final estimate matches the actual carrier exactly.

[11] *Formal counterpart:* `Tier30.Research.NavierStokes.Evolution.periodicH2_localHadamardWellPosed`.

### 3.3 Maximal history assembly

Restriction, restart, overlap uniqueness, and gluing assemble local trajectories into a maximal history. The maximal lifespan

$$T_{\max} = \sup\{T > 0 : \text{a native } H^2 \text{ solution exists through } T\} \tag{23}$$

is derived from the local existence surface.

Maximality therefore expresses an extension boundary. Equation (5) then follows from the uniform restart theorem and carries the full eventual divergence statement.

## 4 The exact classical/native bridge

A breakdown theorem stated in classical variables can enter the native continuation theory only through an exact data bridge. This section records that bridge.

### 4.1 Classical periodic initial data

Let

$$u_0 : \mathbb{R}^3 \to \mathbb{R}^3$$

be smooth, real, divergence-free, and one-periodic. Periodicity descends $u_0$ to $\mathbb{T}^3$. Smoothness gives every finite Fourier-defined Sobolev order:

$$u_0^{\mathbb{T}} \in H^m(\mathbb{T}^3) \qquad \forall m \in \mathbb{N}. \tag{24}$$

Real-valuedness gives Eq. (20). Classical divergence-free structure gives Eq. (21) after the derivative multiplier is identified.

The native initial-data adapter then satisfies the exact physical round trip

$$u_{0,\mathrm{out}}(x) = u_{0,\mathrm{in}}(x) \qquad \forall x \in \mathbb{R}^3. \tag{25}$$

[12]

### 4.2 The $2\pi$ convention

On the unit torus, the classical Fourier basis satisfies

$$\partial_j \longleftrightarrow 2\pi i k_j, \qquad \Delta \longleftrightarrow -4\pi^2|k|^2.$$

The native spectral operators use

$$ik_j, \qquad -|k|^2.$$

The exact conversion is

$$\tau = 2\pi t, \qquad \nu_{\mathrm{int}} = 2\pi\nu_{\mathrm{cl}}, \tag{26}$$

[12] *Formal counterpart:* `Tier30.Research.NavierStokes.ExternalBridge.comparatorInitial_roundTrip`.

with forcing relation

$$f_{\mathrm{cl}}(x,t) = 2\pi f_{\mathrm{int}}(x, 2\pi t) \tag{27}$$

at native start time zero.

This conversion fixes the identity between the native spectral law and the classical unit-period PDE.

## 4.3 Classical forcing

The comparator force is smooth on the future half-space and periodic in space. A smooth extension supplies a globally typed native forcing curve. After the normalization in Eqs. (26)–(27), classical reconstruction gives

$$f_{\mathrm{out}}(x,t) = f_{\mathrm{in}}(x,t) \qquad (t \geq 0). \tag{28}$$

[13]

The force therefore returns to the exact comparator datum on its physical time domain.

## 4.4 All-orders bootstrap and pressure

Smooth initial data and smooth forcing propagate every finite Sobolev order on every finite native time window. Strong time derivatives lift through the Sobolev hierarchy. Smooth representatives are constructed and their derivatives are identified with the classical Fréchet derivatives.

The native projected equation

$$\partial_t u + \mathbb{P}\nabla\cdot(u\otimes u) = \nu\Delta u + \mathbb{P}f \tag{29}$$

recovers a mean-zero periodic pressure from the Leray complement. The resulting smooth physical representative satisfies

$$\partial_t u + (u\cdot\nabla)u = \nu\Delta u - \nabla p + f. \tag{30}$$

An infinite native maximal trajectory therefore produces a global smooth classical periodic solution for the same external initial field and forcing.[14]

Combining this construction with the round-trip equalities yields

$$\boxed{\text{native global trajectory} \Longrightarrow \text{global classical solution for the original comparator datum.}} \tag{31}$$

[15]

Equation (31) is the exact contradiction surface behind the implication diagram in Eq. (1.2).

[13] *Formal counterpart:* `Tier30.Research.NavierStokes.ExternalBridge.comparatorForce_roundTrip`.

[14] *Formal counterpart:* `Tier30.Research.NavierStokes.ClassicalBridge.globalH2Strong_implies_globalClassicalSmoothPeriodic`.

[15] *Formal counterpart:* `Tier30.Research.NavierStokes.ExternalBridge.internalGlobalSolution_implies_comparatorGlobalSolution`.

## 5 Classical breakdown data imply finite maximal lifespan

Let $\nu > 0$. Let $u_0$ be smooth, real, divergence-free, and one-periodic, and let $f$ be smooth, one-periodic, and satisfy the declared future-time comparator regularity. Assume

$$\neg\exists(u,p)\ \mathsf{GlobalSmoothPeriodicNS}(\nu, u_0, f; u, p). \tag{32}$$

The exact bridge of Section 4 produces a physical native initial datum and smooth native scenario. If the native maximal lifespan were infinite, Eq. (31) would construct a global smooth classical solution for the original initial field and force, contradicting Eq. (32). Therefore

$$T_{\max} < \infty. \tag{33}$$

[16]

The finite-breakdown mechanism proved in Section 2 now applies in full. In particular, the maximal $H^2$ trajectory has eventual $H^2$ divergence, bounded kinetic energy, high-frequency $H^2$ escape above every fixed cutoff, bounded $H^2$ mass at every fixed positive heat scale, failure of uniform fine-scale control, and collapse of every uniform positive bounded-control continuation certificate.

This implication is independent of the source of the breakdown datum. Any future periodic construction satisfying the same classical interface inherits the same continuation anatomy.

## 6 Application to the announced forced breakdown theorem

OpenAI announced on September 8, 2026 that it had obtained an analytical and Lean-formalized proof of finite-time Navier–Stokes breakdown and released both the manuscript and formal repository [5, 6, 7]. OpenAI states that the result establishes alternatives (C) and (D) in Fefferman's official formulation.

The periodic theorem exposed by the public formalization has the form

$$\forall \nu > 0, \quad \exists u_0, f : \mathsf{InitPeriodic}(u_0) \wedge \mathsf{ForcePeriodic}(f) \wedge \neg\mathsf{GlobalPeriodicSolutionExists}(\nu, u_0, f). \tag{34}$$

The corresponding Lean declaration is

`NavierStokes.ComparatorBridge.navier_stokes_breakdown_periodic`.

The underlying periodic construction contains additional structure. The initial velocity may be chosen as

$$u_0 = 0, \tag{35}$$

the force is smooth and has compact future time support, and the pre-singular velocity satisfies

$$\forall M > 0,\ \forall \delta > 0, \quad \exists t, x :\ 0 < t < 1, \quad 1 - \delta < t, \quad M < |u(t,x)|. \tag{36}$$

[16] *Formal counterpart:* `Tier30.Research.NavierStokes.ExternalBridge.stableComparatorBreakdown_implies_finiteMaximalLifespan`.

The public Lean project reports the theorem with zero `sorry`. An isolated verifier used for the present work rebuilds the exact external theorem at the audited OpenAI commit and reports the axiom surface

$$[\texttt{propext}, \texttt{Classical.choice}, \texttt{Quot.sound}],$$

with zero `sorryAx`. The present repository retains its own Lean/Mathlib pin, so the external theorem enters the local development as an explicit proposition rather than an imported cross-version proof term.

Assuming mathematical acceptance of Eq. (34), the breakdown bridge in Section 5 applies directly. For every positive viscosity the announced periodic datum therefore has a corresponding native trajectory with finite maximal lifespan, followed by the complete finite-breakdown mechanism of Section 2. The OpenAI construction supplies the witness; the continuation anatomy supplies the native description of its finite maximal failure.

The direct pointwise observable in Eq. (36) is also compatible with the $H^2$ mechanism. The native periodic Sobolev embedding gives

$$|u(x)| \le C_{\mathrm{emb}} \|u\|_{H^2}. \tag{37}$$

Thus arbitrarily large pointwise speed near the designated endpoint forces arbitrarily large $H^2$ values near that endpoint.[17]

The full eventual $H^2$ convergence in Eq. (5) follows from finite maximality and the restart theorem. It carries stronger temporal content than the direct pointwise unboundedness statement.

Under the native clock convention, classical time 1 maps to

$$\tau = 2\pi. \tag{38}$$

The generic comparator theorem supplies global classical nonexistence. Exact equality

$$T_{\max} = 2\pi$$

belongs to a richer candidate-level transport in which the pre-singular solution itself is identified with the native maximal trajectory. The present application preserves the distinction and uses only the authority already carried by the comparator datum.

# 7 Clay status and the zero-force question

The official Clay formulation is essential to the interpretation.

For the periodic positive branch, alternative (B) asks for global smoothness with zero force:

$$\forall u_0 \in \mathsf{Data}_0, \qquad \exists (u,p)\ \mathsf{GlobalSmoothPeriodicNS}(\nu, u_0, 0; u, p). \tag{39}$$

The periodic breakdown branch (D) permits smooth admissible forcing:

$$\exists u_0 \in \mathsf{Data}_0,\ \exists f \in \mathsf{Force}_{\mathrm{Clay}}, \qquad \neg\exists (u,p)\ \mathsf{GlobalSmoothPeriodicNS}(\nu, u_0, f; u, p). \tag{40}$$

[17] *Formal counterpart:* `Tier30.Research.NavierStokes.ExternalBridge.speedUnboundedAtOne_implies_H2UnboundedAtOne`.

The force quantifier creates two distinct mathematical statements. Their logical relation is

$$\boxed{(D) \not\Rightarrow \neg(B).} \tag{41}$$

Global regularity for every smooth zero-force periodic datum can coexist logically with breakdown for some smooth forced periodic datum.

Fefferman's official problem accepts any one of alternatives (A)–(D) as a mathematical resolution route [1]. OpenAI's announcement states that its theorem establishes (C) and (D) [5]. If the announced theorem receives mathematical acceptance, either alternative resolves the mathematical problem as posed.

The institutional prize status follows Clay's separate procedure. As of September 13, 2026, Clay continues to list Navier–Stokes among its unsolved Millennium Prize Problems [2, 3]. Clay's rules require a qualifying publication, a minimum of two years after publication, and general acceptance in the global mathematics community before a proposed solution can enter prize consideration [4]. These procedures supply the current public status while mathematical review proceeds.

The zero-force question therefore remains sharply stated:

$$f = 0: \qquad \text{does every smooth three-dimensional datum remain smooth globally?} \tag{42}$$

The forced theorem and the zero-force theorem occupy adjacent parts of the same equation and distinct parts of the quantifier structure.

# 8 What the continuation anatomy clarifies

The theorem package in Section 2 places several simultaneous features inside one finite-maximal trajectory. Their conjunction carries the interpretation.

## 8.1 Breakdown is a continuation statement

Throughout $[0, T_{\max})$, the trajectory satisfies the Navier–Stokes equation in the declared strong class. The maximal endpoint records exhaustion of bounded strong continuation: every bounded state/forcing control which would provide a uniform restart becomes unavailable near the endpoint. The governing equation remains valid throughout the maximal interval; continuation within the declared strong $H^2$ solution class reaches its terminal boundary.

## 8.2 Kinetic energy and strong regularity separate

Writing $E(t) = \frac{1}{2}\|u(t)\|_{L^2}^2$, kinetic energy remains uniformly finite while the $H^2$ norm diverges. These are simultaneous properties of the same trajectory. The breakdown mechanism therefore lives in derivative-weighted regularity while the coarse kinetic-energy carrier remains finite.

### 8.3 The regularity loss is genuinely fine-scale

For every fixed Fourier cutoff $K$, the $H^2$ mass above $K$ diverges. The theorem says exactly that the divergent $H^2$ mass cannot remain confined below any fixed spectral cutoff. It allows the divergent tail to remain broad and carries no claim of a localized traveling shell or physical-space concentration.

### 8.4 Every fixed positive heat presentation remains controlled

For every fixed $a > 0$, the heat-regularized trajectory remains uniformly $H^2$-controlled. This is an analytic statement about heat-regularized presentations of the trajectory; it is not a claim that a physical measurement apparatus or an independently filtered Navier–Stokes dynamics remains nonsingular.

The scale family therefore carries two simultaneous facts: every declared positive heat scale remains regular, and uniform $H^2$ control disappears as the regularization scale approaches zero. The singular obstruction is located in uniform recovery across vanishing scale.

### 8.5 Continuation capacity is prospective

Every submaximal state carries some positive local continuation certificate under an available force bound. Finite maximality excludes one uniform positive certificate persisting to the endpoint. The trajectory retains local continuation at each submaximal instant while the uniform reserve needed to cross the maximal horizon disappears.

## 9 Questions sharpened by the forced result

The announced theorem and the independent continuation anatomy leave a smaller set of sharper questions.

### 9.1 Unforced regularity

Equation (42) remains the most direct neighboring problem. A forced breakdown theorem shows that positive viscosity together with smooth forcing permits finite classical failure. The zero-force system may possess a different global regularity structure. Establishing or refuting that structure requires a theorem with the force fixed to zero.

### 9.2 Forcing cessation

The richer external construction gives compact future time support for the force. The next decisive timing question is the relation between the forcing support endpoint $T_f$ and the singular endpoint:

$$T_f \ ? \ T_{\mathrm{sing}}. \tag{43}$$

A proof of

$$T_f < T_{\mathrm{sing}} \tag{44}$$

would produce a final autonomous interval ending at the singularity:

$$f(t) = 0 \qquad (T_f \le t < T_{\mathrm{sing}}). \tag{45}$$

The force would then prepare a state whose subsequent autonomous evolution reaches the singular endpoint. This would place the constructed mechanism substantially closer to the historical zero-force question while preserving the distinct issue of which initial states can generate that autonomous trajectory. The ordering in Eq. (43) remains open here.

### 9.3 Robustness

The announced construction supplies existence. A complementary dynamical question asks how that existence sits inside data space. Natural tests include smooth perturbations of the forcing and initial state:

$$(u_0, f) \longmapsto (u_0 + \varepsilon v_0,\ f + \varepsilon g). \tag{46}$$

Persistence under an open neighborhood would identify a robust singular regime. Sensitivity to a thin constraint surface would identify a more finely organized mechanism. Both outcomes carry mathematical content.

### 9.4 Critical continuation criteria

The native theorem uses $H^2$ because it is the strong carrier in which the local evolution is formalized. Classical regularity theory offers sharper continuation criteria involving critical spacetime norms, velocity gradients, or vorticity. Formalizing those criteria would identify which critical quantities must fail along a finite-maximal trajectory and would refine the continuation anatomy below the $H^2$ level.

These four questions now form a compact forward program: zero-force regularity, forcing cessation, robustness, and critical continuation.

## 10 Conclusion

This paper proves that any smooth periodic classical breakdown datum satisfying the declared interface has finite maximal $H^2$ lifespan. Along the associated strong trajectory, the kinetic energy

$$E(t) = \frac{1}{2}\|u(t)\|_{L^2}^2$$

remains uniformly finite while derivative-weighted regularity escapes every fixed spectral cutoff. Every fixed positive heat presentation remains $H^2$-controlled, while uniform fine-scale recovery and every uniform positive bounded-control continuation reserve disappear at the maximal endpoint. These conclusions depend on the breakdown premise itself and therefore remain independent of any particular singular construction.

The announced OpenAI forced breakdown theorem supplies a first concrete application. Its forced Clay alternatives and the zero-force regularity problem carry different quantifiers, so the historical zero-force question remains mathematically distinct. The next focused questions are correspondingly clear: whether forcing ceases before the singular endpoint, whether the singular mechanism persists on an open neighborhood of data, and which sharper critical continuation quantities must fail.

$$\boxed{\begin{array}{c}\text{finite-energy flow}\\ +\ \text{fine-scale } H^2 \text{ escape}\\ +\ \text{fixed-scale regularity}\\ +\ \text{collapse of uniform strong continuation capacity.}\end{array}} \tag{47}$$

That is the continuation anatomy of periodic strong breakdown.

## A Lean theorem map

The main text uses standard periodic PDE language. The table below records the principal formal counterparts in the Lean development.

| Mathematical result | Formal declaration or module |
|---|---|
| Physical periodic Leray projector and Sobolev preservation | `Tier30.Research.NavierStokes.Analysis.PeriodicLeray` |
| Physical $H^2$ product and Fourier-convolution identity | `Tier30.Research.NavierStokes.Analysis.H2Gate` |
| Canonical strong $H^2$ operator package | `Tier30.Research.NavierStokes.Analysis.CanonicalStrongH2` |
| Local periodic $H^2$ Hadamard well-posedness | `Tier30.Research.NavierStokes.Evolution.periodicH2_localHadamardWellPosed` |
| Maximal lifespan and maximal strong trajectory | `Tier30.Research.NavierStokes.Evolution.MaximalEvolution` |
| Finite-lifespan eventual $H^2$ divergence | `Tier30.Research.NavierStokes.Evolution.finiteMaximalLifespan_H2Norm_tendsto_atTop` |
| All-orders Sobolev and time bootstrap | `Tier30.Research.NavierStokes.SmoothBootstrap` |
| Real classical pressure recovery | `Tier30.Research.NavierStokes.ClassicalBridge` |
| Global native strong solution gives a global classical smooth periodic solution | `Tier30.Research.NavierStokes.ClassicalBridge.globalH2Strong_implies_globalClassicalSmoothPeriodic` |
| External smooth periodic initial-data round trip | `Tier30.Research.NavierStokes.ExternalBridge.comparatorInitial_roundTrip` |
| External smooth periodic forcing round trip | `Tier30.Research.NavierStokes.ExternalBridge.comparatorForce_roundTrip` |
| Native global solution gives a global solution for the original comparator data | `Tier30.Research.NavierStokes.ExternalBridge.internalGlobalSolution_implies_comparatorGlobalSolution` |
| Comparator breakdown data give finite native maximal lifespan | `Tier30.Research.NavierStokes.ExternalBridge.stableComparatorBreakdown_implies_finiteMaximalLifespan` |
| Comparator breakdown data give native $H^2$ divergence | `Tier30.Research.NavierStokes.ExternalBridge.stableComparatorBreakdown_implies_H2Blowup` |
| Pointwise speed unboundedness gives $H^2$ unboundedness near the same endpoint | `Tier30.Research.NavierStokes.ExternalBridge.speedUnboundedAtOne_implies_H2UnboundedAtOne` |
| Nonlinear energy cancellation and physical energy identity | `Tier30.Research.NavierStokes.FailureMechanism.EnergyIdentity` |
| Finite maximal lifespan preserves a uniform $L^2$ bound | `Tier30.Research.NavierStokes.FailureMechanism.finiteMaximalLifespan_L2Norm_bounded` |
| High-frequency $H^2$ mass diverges above every fixed cutoff | `Tier30.Research.NavierStokes.FailureMechanism.finiteMaximalLifespan_highFrequencyH2_tendsto_atTop` |
| Every fixed positive heat scale remains $H^2$-controlled | `Tier30.Research.NavierStokes.FailureMechanism.finiteMaximalLifespan_fixedHeatScale_H2_bounded` |
| Uniform fine-scale $H^2$ control fails at finite maximality | `Tier30.Research.NavierStokes.FailureMechanism.finiteMaximalLifespan_not_uniformFineScaleH2Control` |
| Uniform bounded-control continuation reserve collapses | `Tier30.Research.NavierStokes.FailureMechanism.finiteMaximalLifespan_continuationReserve_collapses` |
| Canonical finite-breakdown mechanism package | `Tier30.Research.NavierStokes.FailureMechanism.finiteMaximalLifespan_has_failureMechanism` |
| Stable-side proposition representing the unimported external theorem | `Tier30.Research.NavierStokes.ExternalBridge.BLOCKED_EXTERNAL_TOOLCHAIN` |

The external theorem itself is

`NavierStokes.ComparatorBridge.navier_stokes_breakdown_periodic`

in the OpenAI formalization [7]. The present Lean development records its consequences through an explicit external premise.

# B Formal verification and authority boundary

The periodic continuation theory in this paper is theorem-owned inside the Concordia Lean 4 development. The local PDE theory, maximal lifespan, blow-up alternative, classical reconstruction, external-data adapter, energy identity, high-frequency escape, fixed-scale regularity, and continuation capacity statements all compile in the repository's existing toolchain.

The OpenAI Navier–Stokes theorem is maintained in a separate public Lean project [7]. The audited periodic theorem was independently rebuilt at external commit

`f9e8bc5b38b6e212696e8a30e3e91517af887bbd`

with Lean `v4.34.0-rc2` and the exact pinned dependencies of that project. The wrapper theorem's axiom report is

$$[\texttt{propext}, \texttt{Classical.choice}, \texttt{Quot.sound}],$$

with zero `sorryAx`. The present repository preserves its own existing toolchain and records the external theorem as an explicit proposition at the cross-version boundary.

The authority surface is therefore transparent. The theorem

$$\mathsf{BreakdownData} \Longrightarrow T_{\max} < \infty \Longrightarrow \mathsf{FailureMechanism}$$

is internally proved. The existence of a universal periodic breakdown datum for every positive viscosity is attributed to the separately announced and externally verified OpenAI theorem.

This separation keeps the independent continuation mathematics stable through the present period of external review.

## C Statement and status map

| Statement | Status in this paper | Mathematical role |
|---|---|---|
| Clay alternatives (A)–(D) | official external definition | Fefferman's problem specification |
| Periodic breakdown datum $\Rightarrow$ finite native lifespan | internally proved | exact native/classical contradiction bridge |
| Finite native lifespan $\Rightarrow$ $H^2$ divergence | internally proved | maximal strong continuation theorem |
| Finite native lifespan $\Rightarrow$ bounded $L^2$, high-frequency escape, fixed-scale regularity, reserve collapse | internally proved | continuation anatomy |
| Forced periodic breakdown for every $\nu > 0$ | announced and externally verified | OpenAI theorem in its own pinned Lean environment |
| Prize recognition of the announced theorem | pending Clay process | qualifying publication, time, community acceptance, and CMI review |
| Zero-force global regularity | distinct open theorem | alternative (B) carries the zero-force quantifier |
| Generic or perturbatively stable forced breakdown | future dynamical question | robustness in data/forcing space |
| Exact equality $T_{\max} = 2\pi$ for the external candidate | future candidate-level bridge | requires transport of the richer pre-singular trajectory |
| Forward kinetic-energy cascade | separate spectral-transfer theorem | $H^2$ tail escape is regularity-weighted |

## D Formal development and collaboration

The internally proved results are maintained as part of a larger Lean 4 research development. The theorem map above gives the paper-scoped authority surface and keeps project-specific implementation language in the appendix where provenance matters.

The present manuscript is scoped to periodic Navier–Stokes continuation, exact classical/native transport, and the interpretation of finite smooth classical breakdown. Other domains in the repository are logically separate from the theorem package presented here.

Paper-scoped source review, theorem-level verification, and access to relevant formal modules may be arranged in the context of professional collaboration, technical review, or institutional partnership.

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