Critical Regularity Anatomy of Periodic Navier–Stokes Breakdown

Spectral transfer, terminal nonlinear H^2 work, and simultaneous failure of nonendpoint continuation controls

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Abstract

A previous machine-checked continuation theory for smooth forced periodic three-dimensional Navier–Stokes evolution established that any smooth periodic classical breakdown datum gives a native strong trajectory with finite maximal H^2 lifespan, bounded kinetic energy, eventual H^2 divergence, high-frequency regularity escape, fixed positive heat-scale regularity, and collapse of uniform continuation reserve. The present paper resolves the next question: what additional analytic structure must every such finite-maximal trajectory exhibit?

We prove an exact spectral-transfer layer and a hierarchy of physical continuation failures on the same maximal trajectory. The quadratic Navier–Stokes term has zero total kinetic-energy contribution while admitting nontrivial derivative-weighted transfer across finite spectral regions. Smooth forcing has uniformly vanishing high-frequency Sobolev tails on compact time windows, and its derivative-weighted work is separated into viscous absorption plus a vanishing fine-scale remainder. Finite maximality then forces arbitrarily large positive integrated nonlinear H^2 work in finite annuli beyond every prescribed spectral scale, arbitrarily late in the trajectory.

The same trajectory necessarily leaves the gradient and vorticity continuation regimes and every nonendpoint Prodi–Serrin class formalized here. Its physical deformation rate satisfies

$$\|\nabla u(\cdot)\|_{L^\infty} \notin L^1(0, T_{\max}),$$

its physical vorticity satisfies

$$\|\operatorname{curl} u(\cdot)\|_{L^\infty} \notin L^1(0, T_{\max}),$$

its velocity obeys

$$\|u(\cdot)\|_{L^\infty}^2 \notin L^1(0, T_{\max}),$$

and, for every finite $q > 3$,

$$\|u(\cdot)\|_{L^q}^{2q/(q-3)} \notin L^1(0, T_{\max}).$$

The vorticity criterion is obtained through a periodic logarithmic div–curl estimate, physical heat-kernel analysis, and a forced H^3 energy law. The Prodi–Serrin family is obtained from physical torus L^q norms, a periodic $H^1 \to L^6$ realization, exact interpolation, and the critical Young exponent $2q/(q-3)$.

These statements are packaged in a theorem-owned `ClassicalContinuationFailureMechanism`. Finite periodic strong breakdown therefore has one simultaneous critical anatomy: kinetic energy remains finite, regularity escapes every fixed spectral scale, terminal nonlinear regularity work becomes unbounded, accumulated physical deformation and peak vorticity become nonintegrable, and every formalized nonendpoint Prodi–Serrin class fails. The endpoint $L_t^\infty L_x^3$ criterion remains outside the present theorem and marks the next analytic frontier.

1 Introduction

The first paper in this series developed an exact periodic H^2 continuation theory and proved that any smooth periodic classical breakdown datum produces a finite-maximal native strong trajectory whose kinetic energy remains bounded while H^2 regularity escapes every fixed Fourier cutoff [1]. The result identified the continuation boundary. The present paper studies the interior anatomy of that boundary.

Let

$$\partial_t u + (u \cdot \nabla) u = \nu \Delta u - \nabla p + f, \qquad \nabla \cdot u = 0, \qquad \nu > 0, \tag{1}$$

on the unit three-torus $\mathbb{T}^3$. We consider smooth physical initial data and smooth periodic forcing, and write

$$u : [0, T_{\max}) \to H^2(\mathbb{T}^3)$$

for the canonical maximal strong trajectory supplied by the established local and maximal theory.

The upstream continuation theorem gives the premise

$$T_{\max} < \infty. \tag{2}$$

From that premise alone, the earlier theory already yields

$$\sup_{t<T_{\max}} \|u(t)\|_{L^2} < \infty, \qquad \|u(t)\|_{H^2} \longrightarrow \infty, \tag{3}$$

and, for every finite Fourier cutoff K,

$$\mathsf{H}_{2,>K}(u(t)) \longrightarrow \infty. \tag{4}$$

At every fixed heat scale $a > 0$,

$$\sup_{t<T_{\max}} \mathsf{H}_2(e^{a\Delta} u(t)) < \infty. \tag{5}$$

The present paper begins at Eqs. (2)–(5) and asks how the equation itself must realize this fine-scale loss.

The main theorem can be summarized as the following implication surface:

$$\boxed{\begin{gathered} T_{\max} < \infty \\ \Downarrow \\ \text{bounded kinetic energy} \\ + \ H^2 \text{ escape beyond every fixed cutoff} \\ + \text{ fixed positive heat-scale regularity} \\ \Downarrow \\ \text{arbitrarily large terminal nonlinear } H^2 \text{ work} \\ \text{in finite annuli beyond every prescribed spectral scale} \\ \Downarrow \\ \|\nabla u\|_{L^\infty} \notin L^1_t \\ \|\operatorname{curl} u\|_{L^\infty} \notin L^1_t \\ \|u\|^2_{L^\infty} \notin L^1_t \\ \forall q > 3, \quad \|u\|_{L^q}^{2q/(q-3)} \notin L^1_t \\ + \text{ collapse of uniform strong continuation reserve.} \end{gathered}} \tag{1.2}$$

The vertical arrangement in Eq. (1.2) is organizational. The proved statements are simultaneous necessary consequences of finite maximality. No causal ordering between nonlinear spectral work, deformation, vorticity, and Prodi–Serrin failure is asserted.

The theorem package matters for two reasons. First, it refines H^2 divergence into several classical continuation quantities which live much closer to the physical structure of the equation. Second, it connects those continuation failures to an exact finite-mode transfer calculus. The quadratic term conserves total kinetic energy while performing derivative-weighted work across spectral regions. Smooth forcing remains explicitly present and becomes quantitatively negligible in the high-frequency remainder. Finite breakdown therefore combines coarse energetic persistence with increasingly demanding fine-scale nonlinear regularity production.

The formal development is machine-checked in Lean 4. Physical multiplication is connected to Fourier convolution, modewise energy laws are derived from the strong equation, finite-cutoff and annular balances are integrated on exact maximal-history windows, physical derivatives retain the unit-torus 2π normalization, vorticity is related to the physical gradient through an explicitly proved periodic logarithmic div–curl estimate, and the physical torus L^q norms used in the Prodi–Serrin theory are identified with Mathlib's L^p carrier rather than replaced by Fourier coefficient norms.

The endpoint

$$u \in L^\infty_t L^3_x$$

remains outside the present theorem and its exclusion is structural. The energy/interpolation mechanism which closes every $q > 3$ loses its decisive exponent at $q = 3$, and the known endpoint theory belongs to a different compactness and rigidity regime [8]. This boundary will return in Section 11.

1.1 Relation to nearby work

The classical regularity theory of three-dimensional Navier–Stokes contains several families of continuation criteria. Prodi established an early uniqueness criterion in mixed space-time classes [3]. Serrin's interior regularity theorem identified the now-standard scale of velocity integrability [4]. Ladyzhenskaya developed complementary uniqueness and smoothness results for weak Navier–Stokes solutions [5]. At the endpoint, Escauriaza, Seregin and Šverák proved regularity under $L_t^\infty L_x^3$ control using a substantially deeper rigidity argument [8].

The Beale–Kato–Majda theorem was originally established for three-dimensional Euler evolution and identifies time-integrated vorticity supremum as the controlling quantity for smooth breakdown [6]. The viscous forced periodic theorem proved here occupies a neighboring continuation role. Its proof uses a periodic logarithmic div–curl estimate and a forced H^3 energy law to convert integrable physical vorticity supremum into global continuation.

The spectral-transfer component has a different emphasis. The total quadratic contribution to kinetic energy vanishes, while derivative-weighted transfer need not. This distinction is essential in the present setting because the finite-breakdown theorem preserves a bounded L^2 carrier while H^2 mass escapes every fixed cutoff. The formal development therefore keeps kinetic-energy transport and regularity transport as separate objects.

The individual gradient, vorticity, and Prodi–Serrin continuation implications have classical lineage. The present contribution is their machine-checked forced periodic realization on the exact physical carrier, their simultaneous necessity on one finite-maximal trajectory, and their connection to an exact spectral mechanism. In particular, the terminal annular theorem proves that for every maximal-time anchor, every prescribed lower spectral scale, and every target level, some still-higher finite annulus accumulates nonlinear H^2-weighted work beyond that target. This theorem links the multiscale escape established upstream to unbounded terminal nonlinear derivative-weighted work in finite spectral regions.

The announced OpenAI periodic breakdown result provides a concrete external application of the present theorem package, subject to the authority boundary discussed in the first paper [12, 13, 1]. The mathematics developed here remains independent of the source of a breakdown datum. Any smooth periodic datum which enters the established finite-maximal interface inherits the same critical regularity anatomy.

2 The unified finite-breakdown theorem

We first state the complete theorem package before developing its components.

Let a smooth evolution scenario consist of positive viscosity and smooth periodic forcing, and let the initial datum be smooth, real, divergence-free, and periodic. Assume that the canonical maximal H^2 trajectory has finite lifespan:

$$T_{\max} < \infty.$$

2.1 Coarse energy and strong regularity

Define the kinetic energy

$$E(t) = \frac{1}{2}\|u(t)\|_{L^2}^2. \tag{6}$$

Then

$$\sup_{t<T_{\max}} E(t) < \infty, \qquad \|u(t)\|_{H^2} \longrightarrow \infty. \tag{7}$$

For every $K \geq 0$,

$$\mathsf{H}_{2,>K}(u(t)) \longrightarrow \infty. \tag{8}$$

For every fixed $a > 0$,

$$\sup_{t<T_{\max}} \mathsf{H}_2(e^{a\Delta}u(t)) < \infty. \tag{9}$$

These are the upstream multiscale consequences formalized in the first stage of the program.[1]

2.2 Terminal nonlinear regularity work

For integers $0 \leq K < L$, let

$$\mathsf{H}_{2,K<L}(u)$$

denote the H^2 mass in the corresponding finite Fourier annulus, and let

$$\mathsf{W}^{\mathrm{NL}}_{K<L}(s,t)$$

denote the time-integrated nonlinear H^2-weighted work in that annulus.

Then for every maximal-time anchor $s < T_{\max}$, every prescribed lower spectral floor K_0, and every target $M \in \mathbb{R}$, there exist

$$K_0 < K < L, \qquad s < t < T_{\max},$$

such that

$$M < \mathsf{W}^{\mathrm{NL}}_{K<L}(s,t). \tag{10}$$

[2]

Equation (10) is existential in annulus and time. It asserts arbitrarily large positive nonlinear regularity work arbitrarily late and arbitrarily far into frequency. It carries no sign theorem for the kinetic-energy flux and no claim of a monotone traveling cascade front.

2.3 Accumulated physical deformation

Let

$$G(t) = \|\nabla u(t)\|_{L^\infty(\mathbb{T}^3)}$$

[1] *Formal counterpart:* `Tier30.Research.NavierStokes.FailureMechanism.finiteMaximalLifespan_has_failureMechanism`.

[2] *Formal counterpart:* `Tier30.Research.NavierStokes.GradientContinuation.finiteBreakdown_terminalNonlinearRegularityTransfer_unbounded`.

for the actual physical unit-torus representative. Then

$$G \notin L^1(0, T_{\max}). \tag{11}$$

Equivalently, finite maximality excludes

$$\int_0^{T_{\max}} \|\nabla u(t)\|_{L^\infty}\, dt < \infty.$$

[3]

2.4 Accumulated physical vorticity

Let

$$\omega = \operatorname{curl} u, \qquad \Omega(t) = \|\omega(t)\|_{L^\infty(\mathbb{T}^3)}.$$

Then

$$\Omega \notin L^1(0, T_{\max}). \tag{12}$$

[4]

2.5 Velocity-space continuation failure

The $q = \infty$ Prodi–Serrin lane gives

$$\|u(\cdot)\|_{L^\infty}^2 \notin L^1(0, T_{\max}). \tag{13}$$

For every finite real $q > 3$, set

$$p(q) = \frac{2q}{q-3}. \tag{14}$$

Then

$$\|u(\cdot)\|_{L^q}^{p(q)} \notin L^1(0, T_{\max}). \tag{15}$$

[5]

The exponents satisfy

$$\frac{2}{p(q)} + \frac{3}{q} = 1. \tag{16}$$

[3] *Formal counterpart:* `Tier30.Research.NavierStokes.GradientContinuation.finiteBreakdown_physicalGradientSup_not_integrable`.

[4] *Formal counterpart:* `Tier30.Research.NavierStokes.BKM.finiteBreakdown_physicalVorticitySup_not_integrable`.

[5] *Formal counterpart:* `Tier30.Research.NavierStokes.Serrin.finiteBreakdown_serrinControl_not_integrable`.

2.6 Canonical package

These statements, together with bounded kinetic energy, high-frequency H^2 escape, fixed-scale heat regularity, unbounded terminal nonlinear H^2 work, and collapse of continuation reserve, are assembled in one theorem-owned structure:

`ClassicalContinuationFailureMechanism`.

[6]

Thus finite periodic strong breakdown simultaneously leaves the physical gradient and vorticity continuation regimes and every nonendpoint Prodi–Serrin class formalized in the present development.

3 Spectral transfer and derivative-weighted regularity

The first refinement of the continuation anatomy is spectral. The point is to distinguish total kinetic-energy conservation by the quadratic term from its capacity to redistribute derivative-weighted regularity.

3.1 Modewise kinetic-energy balance

For a Fourier mode $k \in \mathbb{Z}^3$, define the mode kinetic energy

$$E_k(u) = \frac{1}{2}|\widehat{u}(k)|^2.$$

The strong equation gives the exact differential law

$$\frac{d}{dt}E_k(u(t)) = T_k(u(t)) - \nu D_k(u(t)) + F_k(u(t), f(t)), \tag{17}$$

where the sign convention is fixed by the native normalized Fourier carrier. For velocity fields a, b, write

$$\langle a, b\rangle_k := \operatorname{Re}\left\langle \widehat{a}(k), \widehat{b}(k)\right\rangle_{\mathbb{C}^3}. \tag{18}$$

Let

$$\mathcal{N}(u) = \mathbb{P}\nabla_{\mathrm{nat}} \cdot (u \otimes u)$$

denote the projected quadratic response, where the native derivative multiplier is ik. Then

$$T_k(u) := -\langle \mathcal{N}(u), u\rangle_k, \tag{19}$$

$$D_k(u) := |k|^2\langle u, u\rangle_k = |k|^2|\widehat{u}(k)|^2, \tag{20}$$

$$F_k(u, f) := \langle f, u\rangle_k = \operatorname{Re}\left\langle \widehat{f}(k), \widehat{u}(k)\right\rangle_{\mathbb{C}^3}, \tag{21}$$

with $\widehat{f}(k)$ denoting the projected native forcing coefficient used by the strong equation. Thus positive T_k means nonlinear kinetic-energy gain by mode k.[7] [8] [9] [10]

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[6] *Formal counterpart:* `Tier30.Research.NavierStokes.Serrin.finiteBreakdown_has_classicalContinuationFailureMechanism`.

[7] *Formal counterpart:* `Tier30.Research.NavierStokes.ForcingTransfer.modeNonlinearEnergyTransfer`.

[8] *Formal counterpart:* `Tier30.Research.NavierStokes.ForcingTransfer.modeViscousDissipation`.

[9] *Formal counterpart:* `Tier30.Research.NavierStokes.ForcingTransfer.modeForcingInjection`.

[10] *Formal counterpart:* `Tier30.Research.NavierStokes.ForcingTransfer.strongSolution_modeEnergy_derivative`.

For every finite mode set S,

$$\frac{d}{dt}\sum_{k\in S}E_k=\sum_{k\in S}T_k-\nu\sum_{k\in S}D_k+\sum_{k\in S}F_k. \tag{22}$$

The low-mode specialization uses genuine Euclidean lattice balls rather than an abstract enumeration of Fourier indices.[11]

3.2 Total nonlinear kinetic-energy cancellation

For physical real divergence-free states,

$$\sum_{k\in\mathbb{Z}^3}T_k(u)=0. \tag{23}$$

[12]

The convolution kernel used in Eq. (23) is the same kernel obtained from the physical product. Its summability and reconstruction are theorem-owned. Local cancellation is proved on the reality-completed involution orbit selected by the complex Fourier representation. The proof therefore follows the actual carrier instead of imposing a bare three-mode identity from outside the formal system.

Equation (23) gives a precise role to the nonlinearity. It contributes no net kinetic energy while remaining free to redistribute energy between modes.

3.3 Kinetic-energy flux

For a cutoff K, define the signed kinetic-energy flux across K by the negative nonlinear transfer into the low-mode region. The exact low/high balances then contain this quantity with opposite signs.

No positivity theorem is proved for this flux. Finite breakdown therefore carries no theorem-owned statement of a forward kinetic-energy cascade. The regularity theorem developed below is a different object.

4 H^2-weighted annular budgets

Define the modewise H^2 mass

$$\mathsf{h}_2(k,u)=(1+|k|^2)^2|\widehat{u}(k)|^2. \tag{24}$$

For a finite annulus

$$A_{K,L}=\{k:K<|k|\le L\},$$

write

$$\mathsf{H}_{2,K<L}(u)=\sum_{k\in A_{K,L}}\mathsf{h}_2(k,u). \tag{25}$$

[11] *Formal counterpart:* `Tier30.Research.NavierStokes.ForcingTransfer.lowModeKineticEnergy_balance`.

[12] *Formal counterpart:* `Tier30.Research.NavierStokes.ForcingTransfer.nonlinearEnergyTransfer_total_zero`.

4.1 Exact finite-annulus balance

Write

$$w_k = 1 + |k|^2$$

for the native inhomogeneous Sobolev weight. The H^2-weighted mode quantities are

$$T_k^{(2)}(u) := w_k^2 T_k(u), \tag{26}$$

$$D_k^{(2)}(u) := w_k^2 D_k(u), \tag{27}$$

$$F_k^{(2)}(u, f) := w_k^2 F_k(u, f). \tag{28}$$

For integers $K < L$, the formal annulus is the open–closed lattice region

$$A_{K,L} = \left\{ k \in \mathbb{Z}^3 : K^2 < |k|^2 \leq L^2 \right\}. \tag{29}$$

The central integrated quantities are therefore

$$\mathsf{W}^{\mathrm{NL}}_{K<L}(s,t) := \int_s^t \sum_{k \in A_{K,L}} T_k^{(2)}(u(r))\, dr, \tag{30}$$

$$\mathsf{D}_{2,K<L}(u) := \sum_{k \in A_{K,L}} D_k^{(2)}(u), \tag{31}$$

$$\mathsf{W}^{f}_{K<L}(s,t) := \int_s^t \sum_{k \in A_{K,L}} F_k^{(2)}(u(r), f(r))\, dr. \tag{32}$$

13 14 15

Every finite annulus satisfies an exact pointwise balance and an exact time-integrated balance:

$$\begin{aligned} \mathsf{H}_{2,K<L}(u(t)) - \mathsf{H}_{2,K<L}(u(s)) + 2\nu \int_s^t \mathsf{D}_{2,K<L}(u(r))\, dr \\ = 2\mathsf{W}^{\mathrm{NL}}_{K<L}(s,t) + 2\mathsf{W}^{f}_{K<L}(s,t). \end{aligned} \tag{33}$$

16

This frequency balance is finite-dimensional and is stated using only finite Fourier sums. Global H^3 dissipation and unrestricted differentiated Fourier sums remain outside its hypotheses.

4.2 Smooth forcing becomes negligible in the fine tail

Smooth periodic forcing has uniformly vanishing high-frequency Sobolev tails on compact time intervals. Its H^2-weighted work obeys a modewise Young estimate against viscosity, leaving a remainder whose high-frequency tail also vanishes uniformly:

$$\sup_{r \in [s,t]} \mathsf{R}^{f}_{>K}(r) \longrightarrow 0 \qquad (K \to \infty). \tag{34}$$

[13] *Formal counterpart:* `Tier30.Research.NavierStokes.ForcingTransfer.annularNonlinearH2Work`.

[14] *Formal counterpart:* `Tier30.Research.NavierStokes.ForcingTransfer.annularH2ViscousDissipation`.

[15] *Formal counterpart:* `Tier30.Research.NavierStokes.ForcingTransfer.annularH2ForcingWork`.

[16] *Formal counterpart:* `Tier30.Research.NavierStokes.ForcingTransfer.annularH2Mass_integratedBalance`.

[17]

The force remains part of the equation and part of the balance. Equation (34) isolates its direct derivative-weighted contribution at sufficiently fine scales.

4.3 Finite annuli realize high-frequency escape

The upstream high-frequency theorem gives

$$\mathsf{H}_{2,>K}(u(t)) \to \infty$$

for every fixed K. Absolute summability of each individual H^2 state implies that a sufficiently large infinite tail is already witnessed by a finite annulus. Hence for every lower cutoff K, every level M, and every sufficiently late terminal region, there exist t and $L > K$ such that

$$M < \mathsf{H}_{2,K<L}(u(t)). \tag{35}$$

[18]

The theorem carries no localization claim beyond the finite annulus itself. The divergent tail may remain broad.

4.4 Terminal nonlinear regularity work

Fix an anchor $s < T_{\max}$ and a target M. Choose K sufficiently large that the anchor state's H^2 tail is small and the forcing remainder above K is small on the finite terminal window. Then choose a later time and finite annulus whose H^2 mass is sufficiently large by Eq. (35). Applying Eq. (33) on the exact maximal-history window yields

$$M < \mathsf{W}^{\mathrm{NL}}_{K<L}(s,t).$$

Thus:

$$\boxed{\begin{gathered}\forall s < T_{\max},\ \forall K_0,\ \forall M,\\ \exists K_0 < K < L,\ \exists t \in (s, T_{\max}) : \quad M < \mathsf{W}^{\mathrm{NL}}_{K<L}(s,t).\end{gathered}} \tag{36}$$

[19]

The nonlinear term can therefore conserve total kinetic energy and simultaneously perform unbounded derivative-weighted work in arbitrarily high-frequency finite spectral regions.

[17] *Formal counterpart:* `Tier30.Research.NavierStokes.ForcingTransfer.smoothForcing_remainder_uniformlyVanishing`.

[18] *Formal counterpart:* `Tier30.Research.NavierStokes.GradientContinuation.finiteBreakdown_has_late_arbitrarilyLargeFiniteAnnulus`.

[19] *Formal counterpart:* `Tier30.Research.NavierStokes.GradientContinuation.finiteBreakdown_terminalNonlinearRegularityTransfer_unbounded`.

5 Physical deformation as a continuation control

The spectral theorem identifies an internal transfer mechanism. The next layer asks which physical continuation quantity must fail.

5.1 Physical derivative normalization

On the unit torus,

$$\partial_j \longleftrightarrow 2\pi i k_j.$$

The physical integer-order derivative energy is therefore

$$E_2^{\mathrm{phys}}(u) = \|u\|_2^2 + \sum_j \|\partial_j u\|_2^2 + \sum_{j,\ell} \|\partial_j \partial_\ell u\|_2^2. \tag{37}$$

Its exact Fourier weight is

$$1 + (2\pi)^2 |k|^2 + (2\pi)^4 |k|^4.$$

The development proves explicit two-sided comparison with the native H^2 mass.[20]

5.2 Integer H^2 commutator

For $|\alpha| \le 2$,

$$D^\alpha[(u \cdot \nabla)u] = u \cdot \nabla D^\alpha u + [D^\alpha, u \cdot \nabla]u.$$

Periodicity and incompressibility give

$$\langle u \cdot \nabla D^\alpha u, D^\alpha u \rangle = 0. \tag{38}$$

The remaining commutator terms obey

$$\left| \sum_{|\alpha| \le 2} \langle D^\alpha[(u \cdot \nabla)u], D^\alpha u \rangle \right| \le C_{\mathrm{comm}} \|\nabla u\|_\infty E_2^{\mathrm{phys}}(u). \tag{39}$$

[21]

Viscosity contributes nonpositive derivative energy, while smooth forcing is controlled by Young's inequality. Thus along a smooth strong window,

$$\frac{d}{dt} E_2^{\mathrm{phys}}(u(t)) \le (2C_{\mathrm{comm}} \|\nabla u(t)\|_\infty + 1)\, E_2^{\mathrm{phys}}(u(t)) + C_f \|f(t)\|_{H^2}^2. \tag{40}$$

[22]

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[20] *Formal counterpart:* `Tier30.Research.NavierStokes.GradientContinuation.physicalH2DerivativeEnergy_equiv_nativeH2`.

[21] *Formal counterpart:* `Tier30.Research.NavierStokes.GradientContinuation.integerH2ConvectionWork_le_gradientSup`.

[22] *Formal counterpart:* `Tier30.Research.NavierStokes.GradientContinuation.physicalH2Energy_derivative_le_gradientSup`.

5.3 Continuation theorem

If

$$\int_0^{T_{\max}} \|\nabla u(t)\|_\infty \, dt < \infty, \tag{41}$$

then Grönwall applied to Eq. (40), together with finite-window smooth forcing control, gives a uniform H^2 bound below any hypothetical finite maximal endpoint. The established restart theorem then extends the solution.

Hence

$$\boxed{\int_0^{T_{\max}} \|\nabla u(t)\|_\infty \, dt < \infty \Longrightarrow T_{\max} = \infty.} \tag{42}$$

[23]

The contrapositive is Eq. (11).

6 Physical vorticity and logarithmic div–curl control

The gradient criterion still measures all first derivatives. Vorticity gives a sharper rotational control.

6.1 Physical curl and Fourier reconstruction

Let

$$\omega = \nabla \times u.$$

The physical Fourier coefficient is

$$\widehat{\omega}(k) = 2\pi i \, k \times \widehat{u}(k). \tag{43}$$

For divergence-free velocity and $k \neq 0$, the vector triple-product identity gives the exact Biot–Savart reconstruction

$$\widehat{u}(k) = \frac{i}{2\pi |k|^2} k \times \widehat{\omega}(k), \tag{44}$$

with the sign fixed by the formal cross-product convention.[24]

The constant velocity mode is not reconstructed from vorticity. Its spatial gradient vanishes, so it does not enter the continuation quantity.

6.2 The logarithmic endpoint

The singular integral relating ω to ∇u does not admit the pointwise bound

$$\|\nabla u\|_\infty \lesssim \|\omega\|_\infty.$$

[23] *Formal counterpart:* `Tier30.Research.NavierStokes.GradientContinuation.periodic_integrableGradientSup_implies_continuation`.

[24] *Formal counterpart:* `Tier30.Research.NavierStokes.BKM.fourierVelocity_of_vorticity`.

The formal development instead proves a periodic logarithmic estimate.

Let $S_t = e^{t\Delta_{\mathrm{phys}}}$, whose Fourier multiplier is

$$e^{-4\pi^2 t|k|^2}.$$

For $0 < \delta \le 1$, decompose

$$\nabla u = \nabla S_1 u + \nabla(S_\delta - S_1)u + \nabla(I - S_\delta)u. \tag{45}$$

The middle term is represented by second derivatives of the periodized Gaussian convolved with physical vorticity. The kernel satisfies

$$\|\nabla^2 K_t\|_{L^1(\mathbb{T}^3)} \le C_H t^{-1}, \qquad 0 < t \le 1. \tag{46}$$

The integral

$$\int_\delta^1 \frac{dt}{t} = \log\frac{1}{\delta}$$

produces the logarithm. The high-frequency residual obeys a positive-power estimate in δ controlled by H^3.

Optimizing δ gives

$$\boxed{\|\nabla u\|_\infty \le C_{\mathrm{BKM}}\,(1 + \|\omega\|_\infty)\,(1 + \log(e + \|u\|_{H^3}))\,.} \tag{47}$$

[25]

The periodized Gaussian, its Fourier coefficients, its derivative L^1 bounds, and its agreement with the physical heat semigroup are all proved within the development.

6.3 Forced H^3 energy

Define the physical third-order derivative energy

$$E_3^{\mathrm{phys}}(u) = \sum_{|\alpha|\le 3} \|D^\alpha u\|_2^2.$$

It is quantitatively equivalent to the native H^3 norm.[26]

The third-order commutator obeys

$$|\mathcal{N}_3(u)| \le C_3 \|\nabla u\|_\infty E_3^{\mathrm{phys}}(u). \tag{48}$$

The only new interpolation input is a specialized estimate controlling the $D^2u\,D^2u$ term. The resulting forced energy inequality has the form

$$\frac{d}{dt}E_3^{\mathrm{phys}}(u(t)) \le (C_3\|\nabla u(t)\|_\infty + 1)\,E_3^{\mathrm{phys}}(u(t)) + F_3(t), \tag{49}$$

with F_3 integrable on finite time windows for smooth forcing.[27]

[25] *Formal counterpart:* `Tier30.Research.NavierStokes.BKM.periodicBKM_logarithmic_gradient_bound`.
[26] *Formal counterpart:* `Tier30.Research.NavierStokes.BKM.physicalH3DerivativeEnergy_equiv_nativeH3`.
[27] *Formal counterpart:* `Tier30.Research.NavierStokes.BKM.physicalH3Energy_derivative_le_gradientSup`.

6.4 Vorticity continuation

Substituting Eq. (47) into Eq. (49) and applying the logarithmic energy transform yields a Grönwall–Osgood inequality with coefficient

$$1 + \|\omega(t)\|_\infty.$$

Therefore

$$\boxed{\int_0^{T_{\max}} \|\omega(t)\|_\infty \, dt < \infty \Longrightarrow T_{\max} = \infty.} \tag{50}$$

[28]

Finite maximality gives Eq. (12).

7 Prodi–Serrin continuation on the physical torus

We now turn from derivative and vorticity controls to velocity spacetime integrability.

7.1 Physical L^q carrier

For smooth periodic velocity, define

$$|u(x)| = \left(\sum_{j=1}^{3} u_j(x)^2 \right)^{1/2}.$$

For finite $q \geq 1$,

$$\|u\|_{L^q(\mathbb{T}^3)} = \left(\int_{\mathbb{T}^3} |u(x)|^q \, dx \right)^{1/q}. \tag{51}$$

The formal development identifies this custom physical expression with Mathlib's genuine $eLpNorm$ of the canonical physical velocity map. The standard unit-torus volume is proved equal to the normalized Haar measure used in the Fourier development, and

$$\mathrm{vol}(\mathbb{T}^3) = 1.$$

Thus physical L^q analysis and Fourier analysis share one measure normalization.

7.2 The $q = \infty$ lane

The pair

$$(p, q) = (2, \infty)$$

is handled separately. The second-order convection work satisfies

$$|\mathcal{N}_2(u)| \leq C_\infty \|u\|_\infty \|D^2 u\|_2 \|D^3 u\|_2 + C_{\mathrm{low}} E_2(u). \tag{52}$$

[28] *Formal counterpart:* `Tier30.Research.NavierStokes.BKM.periodic_integrablePhysicalVorticitySup_implies_continuation`.

Young absorption yields

$$\frac{d}{dt}E_2(u(t)) \le C_{\nu,\infty}\|u(t)\|_\infty^2 E_2(u(t)) + C_0 E_2(u(t)) + F_2(t). \tag{53}$$

Hence

$$\boxed{\int_0^{T_{\max}} \|u(t)\|_\infty^2 \, dt < \infty \Longrightarrow T_{\max} = \infty.} \tag{54}$$

[29]

7.3 Finite $q > 3$: exponent geometry

Fix

$$3 < q < \infty,$$

and define

$$r(q) = \frac{2q}{q-2}, \qquad \theta(q) = \frac{3}{q}, \qquad p(q) = \frac{2q}{q-3}. \tag{55}$$

Then

$$\frac{1}{q} + \frac{1}{r(q)} + \frac{1}{2} = 1, \qquad \frac{2}{p(q)} + \frac{3}{q} = 1. \tag{56}$$

The Young conjugate exponent which multiplies the velocity norm is

$$\frac{2}{1-\theta(q)} = \frac{2q}{q-3} = p(q). \tag{57}$$

This identity fixes the critical time power exactly.

7.4 Periodic Sobolev realization

The required physical interpolation is

$$\boxed{\|D^2 u\|_{L^{r(q)}} \le C_q \|D^2 u\|_2^{1-3/q} \|D^3 u\|_2^{3/q}.} \tag{58}$$

[30]

The proof localizes a smooth periodic scalar component to Euclidean three-space with a compactly supported cutoff, applies the Euclidean Sobolev inequality, transports the derivative estimates back to the torus, and interpolates between genuine physical L^2 and L^6 norms. The derivative bridge is carried by the actual physical Fréchet derivative of the torus representative.

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[29] *Formal counterpart:* `Tier30.Research.NavierStokes.Serrin.periodic_integrableVelocitySupSq_implies_continuation`.

[30] *Formal counterpart:* `Tier30.Research.NavierStokes.Serrin.secondDerivativeSerrinInterpolation`.

7.5 Critical nonlinear estimate

Hölder with Eq. (56) gives

$$|\mathcal{N}_2(u)| \le K_q \|u\|_{L^q} \|D^2 u\|_{L^{r(q)}} \|D^3 u\|_2. \tag{59}$$

Using Eq. (58),

$$|\mathcal{N}_2(u)| \le K_q \|u\|_q \|D^2 u\|_2^{1-3/q} \|D^3 u\|_2^{1+3/q}. \tag{60}$$

Young's inequality with Eq. (57) gives

$$|\mathcal{N}_2(u)| \le \frac{\nu}{2}\|D^3 u\|_2^2 + C_{\nu,q}\|u\|_q^{2q/(q-3)}\|D^2 u\|_2^2 + C_{\text{low}} E_2(u). \tag{61}$$

[31]

The exponent in Eq. (61) is exactly the Prodi–Serrin exponent.

7.6 Finite-q continuation

The forced H^2 energy inequality becomes

$$E_2'(t) \le C_{\nu,q}\left(1 + \|u(t)\|_q^{p(q)}\right) E_2(t) + F_2(t). \tag{62}$$

[32]

If

$$\int_0^{T_{\max}} \|u(t)\|_q^{p(q)}\, dt < \infty, \tag{63}$$

then smooth forcing and Grönwall give a uniform H^2 bound at a hypothetical finite maximal endpoint. The restart theorem gives continuation. Hence

$$\boxed{3 < q < \infty, \qquad \int_0^{T_{\max}} \|u(t)\|_q^{2q/(q-3)}\, dt < \infty \Longrightarrow T_{\max} = \infty.} \tag{64}$$

[33]

The contrapositive is Eq. (15).

8 One trajectory, many failed continuation channels

The preceding theorems do more than collect independent criteria. They place all of the criteria on one maximal trajectory and one exact strong solution.

[31] *Formal counterpart:* `Tier30.Research.NavierStokes.Serrin.H2ConvectionWork_le_serrin`.

[32] *Formal counterpart:* `Tier30.Research.NavierStokes.Serrin.physicalH2Energy_derivative_le_finiteQSerrin`.

[33] *Formal counterpart:* `Tier30.Research.NavierStokes.Serrin.periodic_integrableSerrinControl_implies_continuation`.

Suppose $T_{\max} < \infty$. Then all of the following statements hold simultaneously:

$$\begin{aligned}
\sup_{t<T_{\max}} E(t) &< \infty, \\
\|u(t)\|_{H^2} &\to \infty, \\
\forall K, \quad \mathsf{H}_{2,>K}(u(t)) &\to \infty, \\
\forall a > 0, \quad \sup_{t<T_{\max}} \mathsf{H}_2(e^{a\Delta}u(t)) &< \infty, \\
\forall s < T_{\max},\ \forall K_0,\ \forall M, \quad \exists K_0 < K < L,\ \exists t > s : M &< \mathsf{W}^{\mathrm{NL}}_{K<L}(s,t), \\
\|\nabla u\|_\infty &\notin L^1_t, \\
\|\omega\|_\infty &\notin L^1_t, \\
\|u\|_\infty^2 &\notin L^1_t, \\
\forall q > 3, \quad \|u\|_q^{2q/(q-3)} &\notin L^1_t.
\end{aligned} \tag{65}$$

The formal structure

`ClassicalContinuationFailureMechanism`

packages Eq. (65) together with collapse of uniform continuation reserve.[34]

This unified statement is the mathematical center of the paper.

8.1 Energy persists

Finite maximality retains a bounded kinetic-energy carrier. The nonlinear term contributes zero total kinetic energy. The singular obstruction therefore occupies derivative-weighted regularity rather than total kinetic-energy divergence.

8.2 Every fixed declared scale remains controlled

Every fixed positive heat scale remains H^2-regular. Every fixed finite low-frequency region remains bounded in the derivative-weighted budget supplied by finite energy. The singular behavior emerges beyond every prescribed spectral scale.

8.3 Nonlinear regularity production becomes terminally unbounded

The forcing remainder becomes uniformly small in the high-frequency tail, while arbitrarily large finite annuli appear arbitrarily late. The exact annular balance therefore forces arbitrarily large positive nonlinear H^2 work in some sufficiently fine finite spectral region.

[34] *Formal counterpart:* `Tier30.Research.NavierStokes.Serrin.finiteBreakdown_has_classicalContinuationFailureMechanism`.

8.4 The formalized nonendpoint continuation controls fail

The physical gradient, physical vorticity, velocity $L_t^2 L_x^\infty$, and every finite Prodi–Serrin class formalized here leave their corresponding continuation regimes. Finite breakdown therefore cannot remain inside any of these theorem-owned nonendpoint control surfaces.

These statements define simultaneous necessary structure. They do not order the phenomena causally. The formal system preserves that distinction.

9 Comparator data and external application

The theorem package above requires only a smooth periodic datum whose corresponding maximal strong trajectory has finite lifespan. It therefore applies to every stable comparator breakdown datum admitted by the exact classical/native bridge developed in the first paper.

For such a datum, the repository proves:

$$\|\nabla u\|_\infty \notin L_t^1,$$
$$\|\omega\|_\infty \notin L_t^1,$$
$$\|u\|_\infty^2 \notin L_t^1,$$

and

$$\forall q > 3, \quad \|u\|_q^{2q/(q-3)} \notin L_t^1.$$

The spectral and continuation-capacity components apply simultaneously.

The stable comparator consequences are theorem-owned inside the present repository and require no cross-version import of an external proof term. The announced OpenAI periodic breakdown theorem supplies a prospective universal source of comparator data for every positive viscosity, subject to the external authority boundary documented in Paper I [12, 13, 1].

The logical separation is useful. The present theorem says:

$$\boxed{\text{breakdown datum} \Longrightarrow \text{critical regularity anatomy.}}$$

The existence theorem for a breakdown datum is a separate proposition.

10 What this anatomy says about breakdown

The combined theorem suggests a compact mathematical description.

Finite periodic strong breakdown is compatible with finite kinetic energy. The equation remains valid throughout the maximal interval. Every fixed positive coarse heat presentation remains controlled. The derivative-weighted state moves beyond every fixed spectral cutoff. Arbitrarily large nonlinear regularity work appears in finite annuli beyond every prescribed spectral scale, arbitrarily late. The accumulated physical deformation becomes nonintegrable. The accumulated peak vorticity becomes nonintegrable. Every nonendpoint Prodi–Serrin velocity class formalized here is left behind.

This description is stronger than any one constituent norm divergence. It expresses one trajectory through several distinct analytic projections:

energy projection: bounded
spectral regularity projection: escaping
nonlinear work projection: terminally unbounded
gradient projection: nonintegrable
vorticity projection: nonintegrable
velocity spacetime projections: all nonendpoint Serrin controls fail.

The projections remain mathematically different. Their coincidence on one maximal trajectory is the content.

11 The remaining endpoint and the direction of Paper III

The nonendpoint continuation program now closes at

$$3 < q \le \infty.$$

The remaining classical endpoint is

$$u \in L^\infty_t L^3_x. \tag{66}$$

The present interpolation argument does not reach Eq. (66). As $q \downarrow 3$, the time exponent

$$p(q) = \frac{2q}{q-3}$$

diverges, and the Young absorption mechanism which drives Section 7 reaches its limiting boundary.

The known endpoint regularity theorem of Escauriaza, Seregin and Šverák belongs to a different analytic regime involving endpoint compactness and rigidity [8]. A formal treatment would require new local objects: parabolic cylinders, scale-invariant local quantities, suitable weak solutions, local energy inequalities, pressure localization, rescaling, compactness, ancient limits, and a uniqueness or rigidity mechanism strong enough to eliminate a nontrivial endpoint blow-up profile.

The forced setting introduces an additional question with direct relevance to the distinction between preparation and terminal dynamics. Under parabolic rescaling around a putative singular point,

$$u_r(x,t) = r\,u(x_* + rx, T_* + r^2 t),$$

the forcing rescales as

$$f_r(x,t) = r^3 f(x_* + rx, T_* + r^2 t).$$

For smooth bounded forcing, this scaling suggests that the force should vanish on compact rescaled regions as $r \downarrow 0$. Establishing that statement inside the formal system would identify an asymptotically autonomous terminal core even when the original physical trajectory remains globally forced.

That is the natural direction of Paper III:

$$\boxed{\text{endpoint rigidity} + \text{blow-up rescaling} + \text{asymptotic autonomy.}}$$

The present paper stops before that new local-analysis program begins.

12 Conclusion

This paper proves a unified critical regularity anatomy for any finite-maximal smooth forced periodic three-dimensional Navier–Stokes trajectory in the established strong H^2 theory.

Kinetic energy remains uniformly finite. The total quadratic kinetic-energy contribution cancels. The H^2 state escapes every fixed Fourier cutoff while every fixed positive heat scale remains controlled. Smooth forcing becomes negligible in the fine derivative-weighted remainder. Arbitrarily large positive integrated nonlinear H^2 work appears in finite annuli beyond every prescribed spectral scale, arbitrarily late.

The same trajectory leaves the physical gradient and vorticity continuation regimes and every nonendpoint Prodi–Serrin class formalized here. Physical deformation is not time-integrable. Physical vorticity is not time-integrable. The velocity fails the $L_t^2 L_x^\infty$ continuation criterion. For every finite $q > 3$, it fails the critical Prodi–Serrin class

$$L_t^{2q/(q-3)} L_x^q.$$

These conclusions are assembled in the machine-checked

`ClassicalContinuationFailureMechanism`.

The endpoint $L_t^\infty L_x^3$ remains a separate rigidity problem. The present theorem therefore closes the nonendpoint continuation anatomy and marks a clean boundary between global continuation analysis and the local terminal geometry required next.

$$\boxed{\begin{array}{c} \text{finite kinetic energy} \\ + \text{ fine-scale } H^2 \text{ escape} \\ + \text{ unbounded terminal nonlinear regularity work} \\ + \text{ nonintegrable deformation} \\ + \text{ nonintegrable peak vorticity} \\ + \text{ failure of every nonendpoint Prodi–Serrin class formalized here.} \end{array}} \tag{67}$$

That is the critical regularity anatomy of periodic strong breakdown.

A Lean theorem map

The main text uses standard periodic PDE language. The table below records the principal formal counterparts in the current Lean development.

Mathematical result	Formal declaration or module
Canonical finite-breakdown mechanism	`Tier30.Research.NavierStokes.FailureMechanism.finiteMaximalLifespan_has_failureMechanism`
Total nonlinear kinetic-energy cancellation	`Tier30.Research.NavierStokes.ForcingTransfer.nonlinearEnergyTransfer_total_zero`
Finite-mode and low-mode kinetic-energy balances	`Tier30.Research.NavierStokes.ForcingTransfer.SpectralCutoffBalance`
Exact finite-annulus H^2 balance	`Tier30.Research.NavierStokes.ForcingTransfer.annularH2Mass_integratedBalance`
Smooth forcing high-frequency remainder control	`Tier30.Research.NavierStokes.ForcingTransfer.smoothForcing_remainder_uniformlyVanishing`
Finite-annulus realization of high-frequency escape	`Tier30.Research.NavierStokes.ForcingTransfer.finiteBreakdown_has_arbitrarilyLargeFiniteAnnulus`
Late finite-annulus escape	`Tier30.Research.NavierStokes.GradientContinuation.finiteBreakdown_has_late_arbitrarilyLargeFiniteAnnulus`
Unbounded terminal nonlinear H^2 work	`Tier30.Research.NavierStokes.GradientContinuation.finiteBreakdown_terminalNonlinearRegularityTransfer_unbounded`
Physical/native H^2 derivative-energy equivalence	`Tier30.Research.NavierStokes.GradientContinuation.physicalH2DerivativeEnergy_equiv_nativeH2`
Integer H^2 gradient commutator	`Tier30.Research.NavierStokes.GradientContinuation.integerH2ConvectionWork_le_gradientSup`
Physical gradient continuation	`Tier30.Research.NavierStokes.GradientContinuation.periodic_integrableGradientSup_implies_continuation`
Finite-breakdown physical gradient nonintegrability	`Tier30.Research.NavierStokes.GradientContinuation.finiteBreakdown_physicalGradientSup_not_integrable`
Fourier div–curl reconstruction	`Tier30.Research.NavierStokes.BKM.fourierVelocity_of_vorticity`
Periodic logarithmic div–curl estimate	`Tier30.Research.NavierStokes.BKM.periodicBKM_logarithmic_gradient_bound`
Physical/native H^3 equivalence	`Tier30.Research.NavierStokes.BKM.physicalH3DerivativeEnergy_equiv_nativeH3`
Forced H^3 gradient energy law	`Tier30.Research.NavierStokes.BKM.physicalH3Energy_derivative_le_gradientSup`
Physical vorticity continuation	`Tier30.Research.NavierStokes.BKM.periodic_integrablePhysicalVorticitySup_implies_continuation`
Finite-breakdown physical vorticity nonintegrability	`Tier30.Research.NavierStokes.BKM.finiteBreakdown_physicalVorticitySup_not_integrable`
Physical L^q carrier and Mathlib L^p bridge	`Tier30.Research.NavierStokes.Serrin.PhysicalLp`
Periodic $H^1 \to L^6$ realization	`Tier30.Research.NavierStokes.Serrin.PeriodicSobolev`
Second-derivative Serrin interpolation	`Tier30.Research.NavierStokes.Serrin.secondDerivativeSerrinInterpolation`
Critical finite-q nonlinear work estimate	`Tier30.Research.NavierStokes.Serrin.H2ConvectionWork_le_serrin`
$q = \infty$ Prodi–Serrin continuation	`Tier30.Research.NavierStokes.Serrin.periodic_integrableVelocitySupSq_implies_continuation`

Mathematical result	Formal declaration or module
Finite-q Prodi–Serrin continuation	`Tier30.Research.NavierStokes.Serrin.periodic_integrableSerrinControl_implies_continuation`
Finite-breakdown Serrin failure for every $q > 3$	`Tier30.Research.NavierStokes.Serrin.finiteBreakdown_serrinControl_not_integrable`
Unified classical continuation-failure mechanism	`Tier30.Research.NavierStokes.Serrin.finiteBreakdown_has_classicalContinuationFailureMechanism`

B Formal verification and authority boundary

The results in this paper are theorem-owned in the Concordia Lean 4 development. The canonical source surfaces are rooted at:

`NavierStokesFailureMechanism00`	finite-breakdown multiscale anatomy,
`NavierStokesForcingTransfer00`	mode and annular transfer,
`NavierStokesGradientContinuation00`	physical-gradient continuation and terminal work,
`NavierStokesBKM00`	logarithmic div–curl and vorticity continuation,
`NavierStokesSerrin00`	physical Prodi–Serrin continuation.

The central analytic estimates and exported continuation theorems used in this manuscript contain no paper-level axioms standing in for those estimates. The physical gradient criterion, physical vorticity criterion, and Prodi–Serrin criteria are exported without analytic estimate witnesses. Compatibility interfaces retained for repository maintenance are not part of the paper-scoped authority surface.

The paper-scoped source surface was audited at repository revision

`7e66913acea8c62eec7e92ef1df68dfe0d622ee0`.

The Navier–Stokes theorem chain used here is already present at that revision; later unrelated repository development does not enter the manuscript's authority surface.

The external OpenAI theorem is not required by the internal implication

$$T_{\max} < \infty \Longrightarrow \mathsf{ClassicalContinuationFailureMechanism}.$$

Its role is only to provide a possible source of external breakdown data, through the separately audited comparator bridge described in Paper I.

C Statement and status map

Statement	Status	Role
Finite maximal lifespan $\Rightarrow$ bounded kinetic energy and H^2 divergence	proved	upstream continuation anatomy
High-frequency H^2 escape and fixed-scale heat regularity	proved	multiscale failure geometry
Total nonlinear kinetic-energy transfer sums to zero	proved	coarse energetic redistribution law
Arbitrarily large terminal nonlinear H^2 work in finite annuli beyond every prescribed spectral scale	proved	derivative-weighted transfer mechanism
$\int \|\nabla u\|_\infty < \infty \Rightarrow$ continuation	proved	physical deformation criterion
$\int \|\omega\|_\infty < \infty \Rightarrow$ continuation	proved	physical vorticity criterion
$u \in L_t^2 L_x^\infty \Rightarrow$ continuation	proved	$q = \infty$ Prodi–Serrin lane
$u \in L_t^{2q/(q-3)} L_x^q,\ q > 3 \Rightarrow$ continuation	proved	finite-q Prodi–Serrin family
Finite breakdown leaves the formalized gradient, vorticity, and nonendpoint Prodi–Serrin regimes	proved	unified critical regularity anatomy
$u \in L_t^\infty L_x^3 \Rightarrow$ regularity	outside present theorem	endpoint rigidity frontier
Asymptotic disappearance of smooth forcing under singular blow-up rescaling	future theorem target	Paper III asymptotic-autonomy program

D Formal development and collaboration

The theorem package presented here is part of a larger Lean 4 research development. The paper-scoped theorem map above records the authority surface used in the manuscript and keeps repository-specific implementation detail concentrated in the appendices.

The present manuscript is restricted to the critical regularity anatomy of finite periodic Navier–Stokes breakdown. Other formal domains in the repository remain logically separate from this theorem package.

Paper-scoped source review, theorem-level verification, and access to relevant formal modules may be arranged in the context of professional collaboration, technical review, or institutional partnership.

References

[1] Z. James, "Continuation Anatomy of Periodic Navier–Stokes Breakdown: A machine-checked H^2 theory of finite-energy regularity escape," preprint (2026), doi:10.5281/zenodo.22737978.

[2] J. Leray, "Sur le mouvement d'un liquide visqueux emplissant l'espace," Acta Math. **63** (1934) 193–248.

[3] G. Prodi, "Un teorema di unicità per le equazioni di Navier–Stokes," Ann. Mat. Pura Appl. **48** (1959) 173–182, doi:10.1007/BF02410664.

[4] J. Serrin, "On the interior regularity of weak solutions of the Navier–Stokes equations," Arch. Ration. Mech. Anal. **9** (1962) 187–195, doi:10.1007/BF00253344.

[5] O. A. Ladyzhenskaya, "On the uniqueness and on the smoothness of weak solutions of the Navier–Stokes equations," Zap. Nauchn. Sem. LOMI **5** (1967) 169–185.

[6] J. T. Beale, T. Kato and A. Majda, "Remarks on the breakdown of smooth solutions for the 3-D Euler equations," Comm. Math. Phys. **94** (1984) 61–66, doi:10.1007/BF01212349.

[7] L. Caffarelli, R. Kohn and L. Nirenberg, "Partial regularity of suitable weak solutions of the Navier–Stokes equations," Comm. Pure Appl. Math. **35** (1982) 771–831.

[8] L. Escauriaza, G. Seregin and V. Šverák, "$L_{3,\infty}$-solutions of Navier–Stokes equations and backward uniqueness," Russian Math. Surveys **58** (2003) 211–250.

[9] T. Tao, "Finite time blowup for an averaged three-dimensional Navier–Stokes equation," J. Amer. Math. Soc. **29** (2016) 601–674, arXiv:1402.0290 [math.AP].

[10] T. Buckmaster and V. Vicol, "Nonuniqueness of weak solutions to the Navier–Stokes equation," Ann. of Math. **189** (2019) 101–144.

[11] D. Albritton, E. Brué and M. Colombo, "Non-uniqueness of Leray solutions of the forced Navier–Stokes equations," Ann. of Math. **196** (2022) 415–455.

[12] OpenAI, "On the Navier–Stokes Millennium Prize Problem," September 8, 2026, openai.com/index/navier-stokes-solution.

[13] OpenAI, "Finite Time Blowup for Navier–Stokes," 2026, public manuscript.

[14] L. de Moura and S. Ullrich, "The Lean 4 Theorem Prover and Programming Language," in *Automated Deduction – CADE 28*, Lecture Notes in Computer Science **12699**, Springer (2021) 625–635.

[15] The mathlib Community, "The Lean Mathematical Library," in *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs*, ACM (2020) 367–381, arXiv:1910.09336 [cs.LO].