Critical Schwarzschild Scalar Dynamics

A machine-checked operator theory of terminal self-adjoint laws and Friedrichs selection

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Abstract

We develop a machine-checked operator theory for the spherically symmetric massless scalar field in the Schwarzschild interior and classify the self-adjoint radial realizations permitted at the curvature endpoint. Starting from the scalar action in regular ingoing coordinates, we derive the radial wave operator, pass through an exact terminal coordinate, and obtain a Schrödinger-type differential expression on $L^2((0,\infty),dx)$ with

$$\widetilde{V}_M(x) = -\frac{1}{4x^2} + R_M(x), \qquad x^2 R_M(x) \longrightarrow 0 \quad (x \downarrow 0).$$

The coefficient $-1/4$ is the Hardy-critical inverse-square value.

The minimal and maximal differential operators are constructed independently. Weak regularity for second derivatives closes the adjoint problem,

$$K_0^* = K_{\max}, \qquad K_{\min}^* = K_{\max},$$

where $K_{\min} = \overline{K_0}$. The horizon end $x \to \infty$ is limit-point and the curvature end $x \downarrow 0$ is limit-circle for the full Schwarzschild potential. The genuine deficiency spaces therefore satisfy

$$\dim_{\mathbb{C}} \mathcal{N}_+ = \dim_{\mathbb{C}} \mathcal{N}_- = 1,$$

so the radial operator is not essentially self-adjoint and its self-adjoint realizations form a genuine $U(1)$ family.

Every maximal-domain field has a unique terminal trace

$$u(x) = \sqrt{x}\big(A_u + B_u \log x\big) + o(\sqrt{x}),$$

together with the independently controlled derivative asymptotic

$$\sqrt{x}\, u'(x) = \frac{A_u}{2} + B_u + \frac{B_u}{2}\log x + o(1).$$

The maximal Green form is exactly

$$\langle K_{\max} u, v\rangle - \langle u, K_{\max} v\rangle = \overline{B_u} A_v - \overline{A_u} B_v.$$

Zero terminal Green flux does not select one member of the self-adjoint family. The full Schwarzschild potential, however, possesses an exact positive zero-energy solution and an associated nonnegative ground-state form. For every maximal-domain field,

$$\boxed{\mathfrak{q}_M[u] < \infty \quad \Longleftrightarrow \quad B_u = 0.}$$

Requiring this finite full-potential form energy on an entire self-adjoint domain selects exactly the no-log realization. We further prove that the closure of the original compact-core quadratic form is represented by precisely this operator. Hence the selected no-log realization is the Friedrichs extension of the closed minimal Schwarzschild operator:

$$\boxed{K_{\mathrm{F}} = K_{\text{no-log}}.}$$

The selected operator is nonnegative and has trivial zero-energy kernel.

The result is deliberately test-field scoped. It classifies the terminal self-adjoint domain ambiguity and derives a same-operator finite-form selector; it does not claim a quantum-corrected metric, spacetime singularity resolution, or gravitational backreaction. The complete theorem chain is formalized in Lean 4 at a frozen paper-scoped repository revision.

1 Introduction

A classical singularity and a radial self-adjointness problem are not the same mathematical object. Geodesic incompleteness identifies a failure of classical spacetime continuation. A quantum test field asks a different operator-theoretic question: once the classical background is fixed, does the radial wave operator determine a unique self-adjoint realization, or must additional terminal domain data be supplied?

Horowitz and Marolf approached this question by using essential self-adjointness as a diagnostic in operator theory for quantum probes of singular spacetimes [1]. In related settings not globally hyperbolic, Ishibashi and Wald showed that sensible linear dynamics satisfying locality and energy requirements is organized by positive self-adjoint extensions of the relevant spatial operator [2, 3]. Singular inverse-square Hamiltonians provide a complementary operator-theoretic language: at critical coupling, local L^2 behavior can support more than one self-adjoint realization, and the boundary condition becomes part of the operator domain rather than a cosmetic endpoint convention [4, 5].

The Schwarzschild interior presents these questions in a particularly precise shape. In regular ingoing coordinates the horizon is nonsingular, while the areal-radius endpoint $r = 0$ is geometrically terminal. After spherical reduction and an exact terminal-coordinate transform, the scalar wave operator becomes a half-line Schrödinger operator whose singular term is exactly Hardy-critical:

$$-\partial_x^2 - \frac{1}{4x^2} + R_M(x). \tag{1}$$

The critical coefficient predicts that endpoint-domain information may survive quantization, but the comparison equation alone does not determine the actual Schwarzschild deficiency indices. The full operator, its adjoint, both endpoints, and the global L^2 problem must all be controlled.

The main theorem package can be summarized as

$$\boxed{\begin{array}{c} \text{ingoing Schwarzschild scalar action} \\ \Downarrow \\ \text{exact half-line operator } \tau_M = -\partial_x^2 + \widetilde{V}_M \\ \Downarrow \\ K_0^* = K_{\max}, \qquad K_{\min}^* = K_{\max} \\ \Downarrow \\ x = \infty \text{ limit-point} \\ x = 0 \text{ limit-circle} \\ \Downarrow \\ (n_+, n_-) = (1, 1) \\ \Downarrow \\ U(1) \text{ family of self-adjoint terminal laws} \\ \Downarrow \\ u(x) = \sqrt{x}(A + B\log x) + o(\sqrt{x}) \\ \Downarrow \\ \mathfrak{q}_M[u] < \infty \iff B = 0 \iff K_{\text{no-log}} = K_{\text{F}}. \end{array}} \tag{1.1}$$

The first conclusion is negative in a precise sense: the bare Schwarzschild radial test-field operator is not essentially self-adjoint. The bulk differential equation and Hilbert carrier leave one real parameter of terminal self-adjoint domain data unresolved.

The second conclusion is selective. The full Schwarzschild potential has an exact positive zero-energy solution. Its ground-state transform produces a nonnegative form on the same maximal fields, with the exact singular kinetic/potential cancellation retained. Finiteness of that form is equivalent to removal of the logarithmic coefficient B. Applied to the classified self-adjoint family, this criterion selects one actual partial operator: the no-log realization.

The selection statement is intentionally more narrow than several nearby claims. We do identify the no-log operator with the Friedrichs extension by completing the original compact-core quadratic form; no independent positivity label or boundary preference is substituted for that construction. We do not identify the radial spectral parameter with a physical interior Killing time. All subsequent quantum-field and gravitational developments lie outside the frozen theorem surface of the present paper. This preprint does not claim a solved backreacted metric or continuation of the Schwarzschild geometry through $r = 0$.

1.1 Relation to nearby work

The essential-self-adjointness criterion of Horowitz–Marolf asks whether a quantum probe operator is uniquely self-adjoint without extra boundary data [1]. Our operator is not essentially self-adjoint, but the analysis then continues through the exact global deficiency calculation and a distinct finite-form selection theorem.

Ishibashi–Wald show, for static non-globally-hyperbolic spacetimes, under specified dynamical assumptions, that acceptable dynamics is generated by positive self-adjoint extensions [2]. The Schwarzschild interior is not a direct instance of that static theorem. The structural comparison is nevertheless useful: extension data is genuine dynamical data, and positivity or energy conditions can further restrict the allowed family. In standard semibounded extension theory, the Friedrichs extension is the canonical self-adjoint operator represented by the closure of the original quadratic form [6]. A central result here is that the independently derived Schwarzschild ground-state form selects exactly that canonical realization.

For the critical inverse-square model, Dereziński and Richard analyze half-line realizations of

$$-\partial_x^2 + \left(\alpha - \frac{1}{4}\right) x^{-2}$$

and make explicit how boundary parameters enter closed and self-adjoint realizations near the critical regime [4]. The present Schwarzschild problem is not replaced by the pure inverse-square operator: the full remainder R_M is retained in the endpoint construction, global deficiency spaces, physical trace, and ground-state selector.

Self-adjoint extension data also remains active in contemporary black-hole models in quantum gravity. Gielen and Menéndez-Pidal derive singularity-resolution statements from unitarity in unimodular time [7], while Gielen and Ried construct a one-parameter extension family in a canonical Schwarzschild-(A)dS quantization and continue to quantum-corrected metric expectation values [8]. The present theorem addresses a different layer: the geometry remains classical, while the scalar test-field operator is constructed, classified, and selected with full domain control.

Recent quantization programs for the Schwarzschild interior continue to study singularity behavior in materially different frameworks. Kanai analyzes Wheeler–DeWitt dynamics in Ashtekar and Barbero variables with minimal-length effects [9]. Fragolino and Rastgoo develop a relational, gauge-invariant interior quantization with a quantum bounce [10]. These approaches quantize reduced gravitational degrees of freedom; the present paper instead isolates the fixed-background scalar test-field operator and its self-adjoint domain theory.

2 The unified Schwarzschild radial theorem

Fix a positive Schwarzschild mass $M > 0$. Let $\mathcal{H} = L^2((0,\infty), dx; \mathbb{C})$ denote the terminal-coordinate Hilbert space constructed below. Let K_0 be the densely defined compact-core realization of the action-derived terminal differential expression, let

$$K_{\min} = \overline{K_0},$$

and let $K_{\max}$ be the independently defined maximal differential operator.

2.1 Action-owned terminal operator

The spherical scalar action in ingoing coordinates gives a radial equation with no artificial pole at the horizon. After the exact terminal-coordinate and Hilbert transport, its differential expression is

$$\tau_M u = -u'' + \widetilde{V}_M u, \tag{2}$$

with

$$\widetilde{V}_M(x) = -\frac{1}{4x^2} + R_M(x), \qquad x^2 R_M(x) \longrightarrow 0 \quad (x \downarrow 0). \tag{3}$$

[1]

2.2 Adjoint closure

The maximal carrier is not defined as an adjoint by fiat. It is defined through local absolute continuity and the requirement that the literal differential expression belong to L^2. The paper proves

$$\boxed{K_0^* = K_{\max}, \qquad K_{\min}^* = K_{\max}.} \tag{4}$$

[2] The second identity uses a separate generic closure-adjoint theorem.

2.3 Endpoint classification

For every nonreal spectral parameter z,

$$x = \infty \quad \text{is limit-point,} \tag{5}$$

[1] *Formal counterpart:* `QGC08.Operator.terminalDifferentialExpression_critical_decomposition`.

[2] *Formal counterpart:* `QGC12.Weak.m9SchwarzschildAdjointAgreement`.

[3] while

$$x = 0 \quad \text{is limit-circle.} \tag{6}$$

[4] The horizon classification is analytic, using tail L^2 interpolation, Wronskian vanishing, and second-order uniqueness. The curvature classification uses full-potential critical Volterra solutions, not the pure comparison basis alone.

2.4 Deficiency indices and self-adjoint realizations

Define the genuine deficiency submodules

$$\mathcal{N}_+ = \ker(K_{\max} - i), \qquad \mathcal{N}_- = \ker(K_{\max} + i).$$

Then

$$\boxed{\dim_{\mathbb{C}} \mathcal{N}_+ = \dim_{\mathbb{C}} \mathcal{N}_- = 1.} \tag{7}$$

[5] Consequently the closed minimal operator is not essentially self-adjoint. Its graph-level self-adjoint extensions are classified by unitary maps $\mathcal{N}_+ \to \mathcal{N}_-$, and because both deficiency spaces are one-dimensional,

$$\boxed{\{\text{self-adjoint radial realizations}\} \simeq U(1).} \tag{8}$$

[6]

2.5 Physical terminal trace

Every maximal-domain representative has unique coefficients $(A_u, B_u) \in \mathbb{C}^2$ with

$$u(x) = \sqrt{x}\,(A_u + B_u \log x) + o(\sqrt{x}) \qquad (x \downarrow 0). \tag{9}$$

[7] The first derivative carries a matching asymptotic that is proved independently from the local-AC derivative, not by formal differentiation of the $o(\sqrt{x})$ remainder:

$$\boxed{\sqrt{x}\,u'(x) = \frac{A_u}{2} + B_u + \frac{B_u}{2}\log x + o(1).} \tag{10}$$

[8] Equivalently,

$$u'(x) = \frac{A_u + B_u \log x + 2B_u}{2\sqrt{x}} + o(x^{-1/2}).$$

The quotient-level trace is complex linear and surjective. Its kernel is exactly the intrinsic closed-minimal boundary kernel.

The maximal Green form becomes

$$\boxed{\langle K_{\max} u, v\rangle - \langle u, K_{\max} v\rangle = \overline{B_u} A_v - \overline{A_u} B_v.} \tag{11}$$

[9] Thus the abstract $U(1)$ extension parameter has a concrete terminal asymptotic meaning.

[3] *Formal counterpart:* `QGC12.schwarzschildOuterLimitPoint`.
[4] *Formal counterpart:* `QGC12.Endpoint.schwarzschildTerminalLimitCircle`.
[5] *Formal counterpart:* `QGC12.Operator.schwarzschild_deficiency_indices`.
[6] *Formal counterpart:* `QGC12.Operator.circleEquivSchwarzschildSelfAdjointRealizations`.
[7] *Formal counterpart:* `QGC12.Endpoint.maximalTerminal_unique_asymptotic`.
[8] *Formal counterpart:* `QGC12.Endpoint.maximalTerminal_first_asymptotic`.
[9] *Formal counterpart:* `QGC12.Operator.maximalSchwarzschild_greenForm_eq_physicalBoundaryGreenForm`.

2.6 Finite full-potential form selection

Let $\mathfrak{q}_M[u]$ denote the exact Schwarzschild ground-state form defined in Section 8. Then every maximal-domain field satisfies

$$\boxed{\mathfrak{q}_M[u] < \infty \quad \Longleftrightarrow \quad B_u = 0.} \tag{12}$$

[10] If a self-adjoint physical trace law is required to have finite ground-state form energy for every field in its domain, its actual partial operator is exactly the no-log operator. [11]

The original compact-core quadratic form

$$q_0[u] = \langle u, K_0 u\rangle, \qquad u \in D(K_0),$$

is closable and equals the full Schwarzschild ground-state integral on the compact core. Logarithmic endpoint cutoffs have vanishing L^2 and form error, so the completed core form is the finite-form $B = 0$ space. The self-adjoint operator represented by that closed form is therefore literally the no-log operator:

$$\boxed{K_{\mathrm{F}} = K_{\text{no-log}}.} \tag{13}$$

[12] [13] Thus the same independently derived criterion has three equivalent descriptions:

$$\text{finite ground-state form} \Longleftrightarrow B = 0 \Longleftrightarrow \text{Friedrichs realization.}$$

The no-log/Friedrichs realization is nonnegative, and its zero-energy kernel is trivial. No uniform positive spectral gap is claimed. [14] [15]

3 From the ingoing Schwarzschild action to a half-line operator

3.1 Radial action in regular coordinates

In ingoing Eddington–Finkelstein coordinates (v, r), the radial inverse metric is regular at $r = 2M$. Write

$$f(r) = 1 - \frac{2M}{r}.$$

For a spherically symmetric real massless scalar $\phi(v, r)$, the angularly normalized radial scalar action density is

$$\mathcal{L}_M(\phi) = -\frac{r^2}{2}\left(2\,\partial_v\phi\,\partial_r\phi + f(r)(\partial_r\phi)^2\right). \tag{14}$$

[16] The factor r^2 is the sphere-area density, while the radial metric determinant has absolute value one.

[10] *Formal counterpart:* `QGC12.Selection.finiteGroundStateFormEnergy_iff_noLog`.
[11] *Formal counterpart:* `QGC12.Selection.physicalFiniteGroundStateEnergyLaw_operator_eq_noLog`.
[12] *Formal counterpart:* `QGC12.Operator.schwarzschildCompletedCoreForm_eq_top`.
[13] *Formal counterpart:* `QGC12.Operator.schwarzschildFriedrichs_eq_noLog`.
[14] *Formal counterpart:* `QGC12.Operator.physicalNoLogLawOperator_nonnegative`.
[15] *Formal counterpart:* `QGC12.Operator.physicalNoLogLawOperator_quadratic_strictPositive`.
[16] *Formal counterpart:* `QGC06.radialScalarActionDensity_eq_metricReduction`.

Differentiating the same action density with respect to the two field derivatives gives the radial Legendre momenta. Their divergence yields the Euler numerator

$$\begin{aligned}\mathcal{E}_M[\phi] = 2r^2\phi_{vr} + 2r\phi_v + (2r-2M)\phi_r \\ + (r^2 - 2Mr)\phi_{rr}.\end{aligned} \tag{15}$$

[17] At $r = 2M$ this expression is finite. The horizon therefore does not enter the radial action through a coordinate denominator.

Introducing the area field

$$\psi = r\phi$$

separates the geometric r^2 factor from the scalar amplitude and produces the standard radial potential

$$V_M(r) = \frac{2M}{r^3} - \frac{4M^2}{r^4}. \tag{16}$$

3.2 Exact terminal coordinate

For $0 < r < 2M$, define

$$x_M(r) = -r - 2M\log\left(1 - \frac{r}{2M}\right). \tag{17}$$

Then

$$x'_M(r) = \frac{r}{2M-r} > 0. \tag{18}$$

The map is an order isomorphism

$$x_M : (0, 2M) \longrightarrow (0, \infty),$$

with

$$x_M(r) \to 0 \quad (r \downarrow 0), \qquad x_M(r) \to \infty \quad (r \uparrow 2M).$$

Thus $x = 0$ represents the curvature endpoint and $x = \infty$ represents the horizon end.

Let $r_M(x)$ denote the inverse radius. The Jacobian transports the natural weighted radial measure to Lebesgue measure, giving an actual unitary Hilbert equivalence to

$$\mathcal{H} = L^2((0,\infty), dx; \mathbb{C}). \tag{19}$$

3.3 Critical terminal potential

Under the exact coordinate transform, the radial differential expression becomes

$$\tau_M = -\partial_x^2 + \widetilde{V}_M(x), \qquad \widetilde{V}_M(x) = V_M(r_M(x)).$$

Near $r = 0$,

$$x_M(r) = \frac{r^2}{4M} + O(r^3), \tag{20}$$

[17] *Formal counterpart:* `QGC06.radialEuler_owned_by_action_momenta`.

and the transported potential obeys Eq. (3).

The comparison equation

$$-u'' - \frac{1}{4x^2}u = 0$$

has the critical basis

$$u_0(x) = \sqrt{x}, \qquad v_0(x) = \sqrt{x}\log x, \tag{21}$$

with nonzero constant Wronskian. Both comparison branches are locally square-integrable at 0. This motivates the endpoint analysis, but no deficiency dimension is inferred from Eq. (21) alone.

4 Minimal, closed-minimal, and maximal operators

4.1 Compact core and closed minimal operator

Let $\mathcal{D}_0$ denote the dense compact smooth/test core on $(0, \infty)$ used by the formal development. Define

$$K_0 u = \tau_M u, \qquad u \in \mathcal{D}_0.$$

Compact-support Green identity gives

$$\langle K_0 u, v\rangle = \langle u, K_0 v\rangle, \qquad u, v \in \mathcal{D}_0. \tag{22}$$

Hence K_0 is densely defined and symmetric, therefore closable. Its graph closure is the closed minimal operator

$$K_{\min} = \overline{K_0}.$$

4.2 Independent maximal differential domain

The maximal domain is defined by differential regularity rather than by the word "adjoint":

$$\mathcal{D}_{\max} = \left\{ u \in L^2(0, \infty) : u, u' \in AC_{\text{loc}}(0, \infty),\ \tau_M u \in L^2(0, \infty) \right\}. \tag{23}$$

The second derivative is an almost-everywhere representative obtained from local absolute continuity. Equality of L^2 value classes implies pointwise equality of the continuous value representatives, then equality of first derivatives by derivative uniqueness, and equality of second derivatives almost everywhere. The differential image is therefore independent of representative.

Define

$$K_{\max} u = \tau_M u, \qquad u \in \mathcal{D}_{\max}.$$

The compact core embeds into $\mathcal{D}_{\max}$ and the two operators agree there.

4.3 Forward adjoint inclusion

For a compact test function φ and a maximal field u, local absolute-continuity integration by parts gives

$$\langle K_0 \varphi, u\rangle = \langle \varphi, K_{\max} u\rangle. \tag{24}$$

No globally C^2 replacement of the maximal domain is used: the second integration-by-parts step is recovered from the local reconstruction identity for u'.

Thus

$$K_{\max} \subseteq K_0^*.$$

4.4 Weak regularity and the reverse inclusion

Suppose $u, v \in \mathcal{H}$ satisfy

$$\langle K_0\varphi, u\rangle = \langle \varphi, v\rangle \qquad \forall \varphi \in \mathcal{D}_0. \tag{25}$$

Every smooth compact test on a positive interval embeds canonically into the core without changing its first or second derivative. Therefore on every compact interval $[a, b] \subset (0, \infty)$,

$$u'' = \widetilde{V}_M u - v \tag{26}$$

in the distributional sense.

Because $\widetilde{V}_M$ is bounded on compact positive intervals and $u, v \in L^2$, the right-hand side of Eq. (26) lies in local L^2. A one-dimensional weak-derivative argument proves first that zero weak derivative implies an almost-everywhere constant function, then that zero weak second derivative implies an almost-everywhere affine function. Double integration of the actual source therefore produces an explicit local representative $\widehat{u}$ with

$$\widehat{u}, \widehat{u}' \in AC_{\mathrm{loc}}, \qquad \widehat{u}'' = \widetilde{V}_M \widehat{u} - v \quad \text{a.e.}$$

Local representatives agree on overlaps by almost-everywhere equality plus continuity, and their derivatives agree by derivative uniqueness. They patch to a global element of $\mathcal{D}_{\max}$ with

$$K_{\max}\widehat{u} = v.$$

This proves the reverse inclusion and therefore Eq. (4). The generic closure-adjoint identity then gives the same maximal adjoint for $K_{\min}$.

5 Endpoint classification and exact deficiency indices

5.1 The horizon end is limit-point

The inverse radius satisfies

$$r_M(x) \longrightarrow 2M^- \qquad (x \to \infty),$$

and the horizon gap obeys an exponential upper bound of the form

$$0 < 2M - r_M(x) \le 2M \exp\left(-\frac{x}{2M}\right). \tag{27}$$

Consequently

$$\widetilde{V}_M(x) \longrightarrow 0, \qquad \widetilde{V}_M \in L^1((1, \infty)).$$

Let $z \in \mathbb{C} \setminus \mathbb{R}$ and let u solve

$$-u'' + \widetilde{V}_M u = zu$$

on the outer tail. If $u \in L^2$, then the bounded tail coefficient gives $u'' \in L^2$. The local-AC reconstruction identities imply the one-dimensional interpolation statement

$$u, u'' \in L^2(1,\infty) \quad \Longrightarrow \quad u' \in L^2(1,\infty).$$

For two same-energy tail solutions u, v, the Wronskian

$$W[u,v] = uv' - u'v$$

is constant. Since both values and first derivatives are square-integrable on the infinite tail, that constant must vanish. Second-order Cauchy uniqueness then makes any two nonreal L^2 tail solutions linearly dependent. This is the limit-point statement Eq. (5).

5.2 The curvature end is limit-circle

The endpoint remainder is sharpened beyond $x^2 R_M(x) \to 0$ to the weighted estimates needed by a critical Volterra equation. Using the comparison basis Eq. (21) and its Green kernel, one constructs two full-potential solutions for every nonreal z whose leading behavior is respectively

$$u_z(x) = \sqrt{x}\,(1 + o(1)), \qquad v_z(x) = \sqrt{x}\log x\,(1 + o(1)). \tag{28}$$

The full potential, including R_M, enters the Volterra equation. The two solutions have nonzero Wronskian and both belong to $L^2(0,\epsilon)$ because

$$\int_0^\epsilon x\,dx < \infty, \qquad \int_0^\epsilon x(\log x)^2\,dx < \infty.$$

This proves the terminal limit-circle statement Eq. (6).

5.3 From endpoint analysis to $(1,1)$

The horizon limit-point theorem implies that every nonreal global deficiency space has dimension at most one. Nontriviality is not inferred from local square integrability. Instead, a cutoff Green probe rules out simultaneous vanishing of the positive and negative deficiency kernels. Reality of the Schwarzschild operator conjugates one kernel into the other. Hence both are nonzero, and the upper bound becomes exact:

$$\dim_{\mathbb{C}} \mathcal{N}_+ = \dim_{\mathbb{C}} \mathcal{N}_- = 1.$$

This is the decisive operator-theoretic classification. The singular endpoint contributes exactly one real parameter of self-adjoint terminal data; the horizon contributes none.

6 Physical endpoint trace and the $U(1)$ family

6.1 From deficiency coordinates to physical coefficients

The Volterra solutions first provide asymptotic coefficients for homogeneous solutions. Variation of parameters extends the analysis to arbitrary maximal fields. Every $u \in \mathcal{D}_{\max}$ has a unique physical trace (A_u, B_u) satisfying Eq. (9). The trace descends through the L^2 quotient because two maximal representatives of the same L^2 class agree pointwise on the positive half-line. It is complex linear and onto $\mathbb{C}^2$.

The derivative trace is controlled at the same level. The actual local-AC derivative satisfies Eq. (10). This matters because the boundary form is therefore recovered from independently proved value and derivative limits; no derivative is taken through an unspecified $o(\sqrt{x})$ term.

The closed-minimal domain is the intrinsic zero-boundary kernel. Under the proved linear equivalence between intrinsic deficiency coordinates and physical coefficients, it is equivalently the zero physical trace:

$$u \in D(K_{\min}) \quad \Longrightarrow \quad A_u = B_u = 0,$$

with the converse in the actual quotient-level boundary kernel characterization used by the extension theory.

6.2 Canonical Darboux Green form

For two arbitrary maximal fields, local-AC Green identity gives on $0 < a < b$

$$\begin{aligned} &\int_a^b \left(\overline{K_{\max}u}\, v - \overline{u}\, K_{\max}v\right) dx \\ &\qquad = W[u,v](b) - W[u,v](a). \end{aligned} \tag{29}$$

The outer Wronskian tends to zero. At the curvature endpoint, Eqs. (9) and (10) make the cancellation explicit. Writing

$$u \sim \sqrt{x}(A_u + B_u \log x), \qquad u' \sim \frac{A_u + B_u \log x + 2B_u}{2\sqrt{x}},$$

and similarly for v, the logarithmic terms cancel in the Wronskian and one obtains

$$\lim_{x\downarrow 0} W[u,v](x) = \overline{A_u} B_v - \overline{B_u} A_v.$$

With the inner-product convention used here, Eq. (29) yields the global form Eq. (11).

6.3 Actual self-adjoint boundary laws

A nonzero physical trace line can be written

$$aA + bB = 0. \tag{30}$$

It is Hermitian precisely when

$$\overline{a}\, b = \overline{b}\, a. \tag{31}$$

The corresponding physical trace-law operator is densely defined, extends the closed minimal operator, and is equal to its own Hilbert adjoint. The physical projective laws are in two-way correspondence with the deficiency-space $U(1)$ phases, including equality of the actual partial operators rather than a relabeling of candidate boundary data.

Thus the classification has two equivalent forms:

$$\text{unitary map } \mathcal{N}_+ \to \mathcal{N}_- \quad \Longleftrightarrow \quad \text{Hermitian physical line in } (A,B).$$

7 Flux conservation is not a selector

Self-adjointness forces the boundary Green form to vanish within each self-adjoint domain. It follows immediately from Eq. (11) that every actual Hermitian physical law has zero polarized terminal Green flux.

This is not enough to determine the terminal law. In particular, the no-log law

$$B = 0 \tag{32}$$

and the no-power law

$$A = 0 \tag{33}$$

are distinct actual self-adjoint realizations and both satisfy zero terminal Green flux.

Therefore

$$\boxed{\text{self-adjointness + zero terminal Green flux} \not\Longrightarrow \text{unique terminal law.}} \tag{34}$$

This negative result separates conservation from selection. A stronger structure is needed.

8 Full-potential ground-state form and finite-form selection

8.1 Exact positive zero-energy solution

The full transported Schwarzschild potential possesses the exact positive zero-energy solution

$$g_M(x) = r_M(x). \tag{35}$$

It satisfies the literal homogeneous equation

$$-g_M'' + \widetilde{V}_M g_M = 0.$$

At the curvature endpoint,

$$\frac{g_M(x)}{\sqrt{x}} \longrightarrow 2\sqrt{M}, \tag{36}$$

so g_M is a nonzero no-log terminal solution. At the horizon,

$$g_M(x) \to 2M, \qquad g_M'(x) \to 0.$$

8.2 Ground-state quotient and exact form identity

For a maximal field u, define its ground-state quotient formally by

$$y = \frac{u}{g_M}.$$

The literal derivative of this quotient is expressed by the ground-state Wronskian

$$W_g[u] = u g_M' - u' g_M.$$

The corresponding nonnegative density is

$$\mathcal{E}_M[u](x) = g_M(x)^2 \left| \left(\frac{u}{g_M} \right)' (x) \right|^2 = \frac{|W_g[u](x)|^2}{g_M(x)^2}. \tag{37}$$

The exact local balance is not a positivity label imposed on the operator. For every $0 < a \leq b$,

$$\begin{aligned} \int_a^b \mathcal{E}_M[u](x)\, dx = \int_a^b \operatorname{Re}\big(\overline{u(x)}\, \tau_M u(x)\big)\, dx \\ + \operatorname{Re} \mathcal{F}_M[u](b) - \operatorname{Re} \mathcal{F}_M[u](a), \end{aligned} \tag{38}$$

where $\mathcal{F}_M$ is the explicit ground-state boundary charge. [18]

Define the full ground-state form energy by

$$\mathfrak{q}_M[u] = \int_0^\infty \mathcal{E}_M[u](x)\, dx \tag{39}$$

when the nonnegative density is integrable.

The point of Eq. (39) is that the separately singular kinetic and potential pieces are never treated as independent improper integrals. Their exact full-potential cancellation has already occurred in the ground-state identity.

8.3 The logarithmic coefficient is the divergence coefficient

Let (A_u, B_u) be the physical terminal trace. The paper proves the sharp density limit

$$x\, \mathcal{E}_M[u](x) \longrightarrow |B_u|^2 \qquad (x \downarrow 0), \tag{40}$$

under the normalization used by the terminal trace. [19]

If $B_u \neq 0$, then Eq. (40) gives a positive inverse-distance lower bound sufficiently near zero, so

$$\int_0^\epsilon \mathcal{E}_M[u](x)\, dx = \infty.$$

Thus finite form energy requires $B_u = 0$.

Conversely, if $B_u = 0$, the curvature ground-state boundary charge tends to zero. The outer charge also tends to zero for every maximal field in the no-log domain. The differential pairing density is integrable because both the value and image belong to L^2. Passing Eq. (38) to the full half-line therefore gives

$$\mathfrak{q}_M[u] = \operatorname{Re}\langle u, K_{\max} u \rangle < \infty. \tag{41}$$

This proves Eq. (12) in both directions.

[18] *Formal counterpart:* `QGC12.Operator.groundStateFormDensity_intervalBalance`.

[19] *Formal counterpart:* `QGC12.Operator.groundStateFormDensity_scaledEndpointLimit`.

8.4 Unique self-adjoint law under the finite-form criterion

Let L be any Hermitian physical trace line and let K_L denote its actual self-adjoint trace-law operator. Define the law-level finite-form condition by

$$\mathfrak{q}_M[u] < \infty \qquad \forall u \in D(K_L).$$

Because the physical trace is surjective, a Hermitian line contains fields with every trace allowed by that line. Equation (12) therefore forces every accepted trace to have $B = 0$. Hence the line is exactly the no-log trace set and the actual partial operator equals the no-log operator.

This is stronger than selecting a label:

$$\boxed{\begin{array}{c}\text{finite full-potential ground-state form energy}\\ \text{for every field in a self-adjoint domain}\\ \Longleftrightarrow\\ B = 0\\ \Longleftrightarrow\\ K_L = K_{\text{no-log}}.\end{array}} \tag{42}$$

The selector is the explicitly derived full Schwarzschild ground-state form itself. The next subsection identifies the resulting operator with the Friedrichs extension by proving that this form is the closure of the original compact-core quadratic form.

8.5 Identification with the Friedrichs extension

Let

$$q_0[u] = \langle u, K_0 u\rangle, \qquad u \in D(K_0), \tag{43}$$

be the quadratic form of the original compact-core Schwarzschild operator. On compactly supported fields, the ground-state identity has no endpoint contribution and gives

$$q_0[u] = \int_0^\infty \mathcal{E}_M[u](x)\, dx. \tag{44}$$

[20]

The remaining issue is form-core ownership. The finite ground-state form must be shown to be the actual closure of Eq. (43). The proof uses logarithmic endpoint cutoffs adapted to the critical $\sqrt{x}$ behavior. The exact positive Schwarzschild ground state satisfies a near-endpoint bound $g_M(x)^2 \le C_M x$, so the cutoff capacity tends to zero. Concrete compact-core approximants converge both in L^2 and in the ground-state form norm. Combined with closed-minimal graph density and the physical trace decomposition, this proves that the topological completion of the original core form is exactly the finite-form no-log space. [21] [22] [23] [24]

[20] *Formal counterpart:* `QGC12.Operator.schwarzschildCoreForm_actualPairing`.
[21] *Formal counterpart:* `QGC12.Operator.schwarzschildGroundCutoff_capacity_tendsto_zero`.
[22] *Formal counterpart:* `QGC12.Operator.groundCompactErrorSequence_L2_normSq_tendsto_zero`.
[23] *Formal counterpart:* `QGC12.Operator.groundCompactErrorSequence_formParameter_tendsto_zero`.
[24] *Formal counterpart:* `QGC12.Operator.schwarzschildCompletedCoreForm_eq_top`.

Let K_{F} denote the self-adjoint operator represented by this closed semibounded form. Computing the represented unit resolvent and comparing it with the already constructed no-log operator gives equality of the actual partial operators, including domain and action:

$$\boxed{K_{\mathrm{F}} = K_{\text{no-log}}.} \tag{45}$$

[25]

Thus the finite-form selection theorem independently recovers the canonical Friedrichs realization:

$$\boxed{\mathfrak{q}_M[u] < \infty \quad \Longleftrightarrow \quad B_u = 0 \quad \Longleftrightarrow \quad u \text{ lies in the Friedrichs form domain.}} \tag{46}$$

At the operator level, imposing the finite-form criterion on the full self-adjoint law selects K_{F} itself.

9 Positivity of the selected realization

On the no-log domain, both endpoint ground-state charges vanish. Equation (41) becomes

$$\mathrm{Re}\langle u, K_{\text{no-log}} u\rangle = \mathfrak{q}_M[u] \geq 0. \tag{47}$$

The selected self-adjoint operator is therefore nonnegative.

The maximal zero-energy kernel is also trivial. If u solves

$$\tau_M u = 0$$

and belongs to the maximal L^2 domain, its Wronskian with the exact positive ground state is constant. Outer decay forces that constant to vanish, so u is proportional to g_M. Since $g_M(x) \to 2M$ as $x \to \infty$, it is not itself in $L^2(0,\infty)$. Therefore the proportionality constant must be zero.

Hence

$$\ker K_{\text{no-log}} = \{0\}. \tag{48}$$

A generic quadratic-discriminant argument then upgrades nonnegativity to strict quadratic positivity on every nonzero domain vector:

$$u \neq 0 \quad \Longrightarrow \quad \langle u, K_{\text{no-log}} u\rangle > 0. \tag{49}$$

No uniform positive spectral gap is asserted.

10 What the operator classification says about the singularity

The result has two logically distinct layers.

First, the bare test-field operator remains nonunique:

$$(n_+, n_-) = (1, 1).$$

[25] *Formal counterpart:* `QGC12.Operator.schwarzschildFriedrichs_eq_noLog`.

In the operator-theoretic quantum-singularity language associated with Horowitz–Marolf, the local wave equation and Hilbert structure do not by themselves determine a unique self-adjoint realization. The singular endpoint retains one real parameter of self-adjoint domain data.

Second, the complete action-owned radial theory contains more than the bare requirement of self-adjointness. The exact positive zero-energy solution and its full-potential form distinguish the $B = 0$ realization. The Friedrichs theorem clarifies this further: the selected member is the canonical self-adjoint realization obtained by closing the original semibounded compact-core form. The independently derived terminal statement $B = 0$ and the standard Friedrichs construction therefore meet at the same domain and action. The resulting statement is therefore not

"quantization automatically removes the singularity."

It is

"self-adjointness permits a family, while a derived finite-form condition selects one member."

This distinction is of note in modern HEP-TH/QG language. A self-adjoint extension parameter can be read as short-distance boundary information left unresolved by the long-distance differential expression. The present theorem does not eliminate that mathematical family by convention. It first exhibits the family and then proves that one member alone carries finite full-potential ground-state form energy across its entire domain.

The logarithmic structure also makes the critical scaling transparent. Under a formal rescaling $x \mapsto \lambda x$,

$$A + B \log x \mapsto A + B \log \lambda + B \log x.$$

The no-log line $B = 0$ is invariant under this elementary trace action. We do not promote this observation to a boundary renormalization-group theorem in the present paper.

11 The remaining quantum and geometric frontier

The selected operator closes the test-field classification problem addressed here. It does not close the gravitational problem.

Continuum Fock dynamics, quantum stress tensors, and sourced gravitational response are subsequent questions and lie outside this preprint's frozen theorem surface. No result from those later stages is used in the operator classification, derivative trace, finite-form selection, or Friedrichs identification proved here.

The genuinely quantum-gravitational frontier lies one step later. A geometric singularity statement requires a covariantly controlled quantum matter source and a gauge-controlled sourced gravitational response,

$$\langle T_{\mu\nu} \rangle_\Psi \longrightarrow DG_{g_M}[\delta g] = 8\pi G \, \langle T_{\mu\nu} \rangle_\Psi,$$

followed by a new analysis of the perturbed metric and its terminal operator. Only that chain can determine whether the geometry itself acquires a continuation. Controlled state-dependent stress tensors in black-hole interiors are themselves an active QFT-in-curved-spacetime problem; recent two-dimensional work computes vacuum polarization and stress-energy inside eternal and collapsing black-hole geometries [11]. That setting is distinct from the four-dimensional Schwarzschild test-field operator studied here, but it illustrates the separate analytic burden carried by the backreaction step.

Accordingly, the present paper makes no claim about a black-hole/white-hole transition, curvature regularization, a Page curve, a unitary asymptotic black-hole S-matrix, or continuation of the Schwarzschild metric through $r = 0$.

12 Conclusion

The spherically symmetric massless scalar field on the Schwarzschild interior gives an exact Hardy-critical half-line operator. The horizon end is analytically limit-point and the curvature end is limit-circle. The independently constructed minimal and maximal operators satisfy

$$K_0^* = K_{\max}, \qquad K_{\min}^* = K_{\max},$$

and the genuine deficiency indices are

$$\boxed{(n_+, n_-) = (1, 1).}$$

Thus the bare radial quantum problem is not essentially self-adjoint. Its self-adjoint terminal laws form a genuine $U(1)$ family.

The same operator also gives a unique physical terminal trace

$$u(x) = \sqrt{x}\,(A + B\log x) + o(\sqrt{x}),$$

with canonical Green form

$$\overline{B}_u A_v - \overline{A}_u B_v.$$

Zero terminal Green flux does not distinguish one extension.

The full Schwarzschild potential then provides the additional structure that self-adjointness alone lacks. Its exact positive zero-energy solution produces a nonnegative ground-state form whose endpoint density has logarithmic coefficient $|B|^2/x$. Consequently

$$\boxed{\mathfrak{q}_M[u] < \infty \quad \Longleftrightarrow \quad B_u = 0.}$$

Requiring this condition across a self-adjoint domain selects exactly the no-log realization. The same finite-form space is the closure of the original compact-core quadratic form, and the represented operator is precisely the Friedrichs extension:

$$\boxed{K_{\text{no-log}} = K_{\mathrm{F}}.}$$

Thus the independently derived Schwarzschild ground-state criterion selects the canonical Friedrichs realization. The selected operator is nonnegative with trivial zero-energy kernel.

The test-field operator problem is therefore classified rather than erased: the differential operator admits a one-parameter family of self-adjoint terminal laws, while the full-potential ground-state form both identifies one distinguished law and recovers the canonical Friedrichs operator. Whether quantum matter built on that realization changes the Schwarzschild geometry is a subsequent question.

A Lean theorem map

The paper-level theorem package is machine-checked in the Concordia Lean 4 development. The principal source surfaces are:

Mathematical surface	Formal source
Ingoing radial scalar action and action-owned Euler equation	`QGC06.Radial.SchwarzschildRadialReduction`, `QGC06.Radial.RadialWaveOperator`
Exact terminal coordinate and inverse-radius equivalence	`QGC08.Coordinate.ExactEquivalence`
Lebesgue L^2 transport and function-level operator conjugation	`QGC08.Hilbert.UnitaryTransport`, `QGC08.Operator.FunctionConjugation`
Dense compact core, minimal operator, symmetry, graph closure	`QGC09.Operator.MinimalSchwarzschildOperator`, `QGC09.Operator.ClosedMinimalSchwarzschildOperator`
Independent maximal differential domain	`QGC10.Operator.MaximalSchwarzschildOperator`
Weak regularity and minimal/maximal adjoint identification	`QGC12.Weak.AdjointIdentification`, `QGC12.Operator.ClosedMinimalAdjoint`
Outer limit-point analysis	`QGC12.Endpoint.OuterEndpoint`, `Tower.Operator.WronskianUniqueness`, `Tower.Operator.SecondOrderUniqueness`
Critical full-potential Volterra solutions and terminal limit-circle theorem	`QGC12.Endpoint.CriticalVolterra`
Genuine deficiency dimensions $(1, 1)$	`QGC12.Operator.DeficiencyDimension`, `QGC12.Operator.DeficiencyDecision`
Graph-level von Neumann extensions and $U(1)$ classification	`QGC12.Operator.SelfAdjointExtensions`
Physical terminal trace, derivative asymptotic, and Green form	`QGC12.Operator.PhysicalTraceClassification`, `QGC12.Endpoint.DerivativeTrace`, `QGC12.Operator.PhysicalDarbouxGreen`
Friedrichs form completion and no-log operator identification	`Tower.Operator.ClosedFormResolvent`, `Tower.Operator.FriedrichsCoreForm`, `QGC12.Operator.CriticalLogCutoff`, `QGC12.Operator.SchwarzschildEndpointCapacity`, `QGC12.Operator.SchwarzschildFormCore`, `QGC12.Operator.SchwarzschildFriedrichs`
Positive ground state and no-log operator positivity	`QGC12.Operator.PositiveGroundState`, `QGC12.Operator.NoLogNonnegative`, `QGC12.Operator.ZeroEnergyKernel`
Finite full-potential form-energy selection	`QGC12.Operator.GroundStateFormDensity`, `QGC12.Operator.GroundStateFormBalance`, `QGC12.Selection.GroundStateFiniteEnergy`, `QGC12.Selection.GroundStateFiniteEnergySufficiency`, `QGC12.Selection.GroundStateGlobalEnergy`
Paper-level capstone and audits	`QGC12.CompletionStatus`, `QGC12AuthorityAudit`, `QGC12DependencyAudit`, `QGC12PhysicalSelectionAudit`

B Formal verification and authority boundary

The results in this paper are theorem-owned in the Concordia Lean 4 development. The operator carriers are constructed before their classification: the maximal operator is not defined to be the adjoint, the deficiency spaces are genuine kernels rather than finite surrogate carriers, and the self-adjoint extensions are graph-level operator domains rather than projective labels.

The paper-scoped source surface was audited at repository revision

`2d084a45e5bf84e88a44a537abd5b46c59d5f866`.

The paper consumes the M171 derivative trace and M172 Friedrichs identification at this revision together with the earlier operator/selection chain. Results beyond M172 remain outside the manuscript's authority surface.

The paper-level endpoint authority is specifically the full-potential result. The limit-circle theorem uses actual nonreal solutions of the transported Schwarzschild potential, while the deficiency dimensions additionally consume the independently proved outer limit-point theorem. The value $(1, 1)$ is not supplied by the critical comparison basis.

Likewise, the no-log selection result is not a stored boundary preference. The physical trace is derived for every maximal field, the form density is derived from the exact positive zero-energy solution, the inverse-distance obstruction is proved when $B \neq 0$, and sufficiency uses the actual global form balance with both endpoint charges. The law-level theorem identifies equality of the resulting partial operator with the no-log operator.

The Friedrichs strengthening begins again at the original compact core. The compact-core quadratic pairing is identified with the same ground-state integral; logarithmic cutoffs have vanishing endpoint capacity; explicit approximants converge in both L^2 and form norm; and the completed form is represented by an operator whose unit resolvent is computed to equal that of the no-log realization. The resulting equality $K_{\mathrm{F}} = K_{\text{no-log}}$ therefore owns both domain and action.

The Friedrichs identification is paper-scoped authority: it consumes the actual compact-core pairing, endpoint cutoff capacity, L^2/form approximation, topological form completion, and represented resolvent equality. It is not inferred merely from nonnegativity. The paper also consumes the independently proved maximal derivative trace. No downstream quantum-field or gravitational response result is used in the present theorem package.

C Statement and status map

Statement	Status	Role
Ingoing Schwarzschild scalar action $\Rightarrow$ exact radial Euler operator	proved	action ownership
Exact terminal coordinate and Lebesgue L^2 transport	proved	half-line Hilbert realization
$K_0^* = K_{\max}$ and $K_{\min}^* = K_{\max}$	proved	true adjoint identification
Horizon endpoint is limit-point	proved	no horizon boundary parameter
Curvature endpoint is limit-circle for the full potential	proved	terminal boundary freedom
Genuine deficiency indices $(n_+, n_-) = (1, 1)$	proved	non-essential self-adjointness
Actual self-adjoint realizations are classified by $U(1)$	proved	self-adjoint terminal-law family
Every maximal field has unique physical trace (A, B)	proved	terminal asymptotic coordinates
Maximal Green form is $\overline{B}_u A_v - \overline{A}_u B_v$	proved	physical boundary symplectic form
Zero terminal Green flux uniquely selects an extension	proved false	nonselection control
Finite full-potential ground-state form energy $\Leftrightarrow B = 0$	proved	field-level selection theorem
Finite-form law on every domain field selects the no-log partial operator	proved	law-level selection theorem
No-log realization is nonnegative with trivial zero kernel	proved	selected-operator positivity
Derivative endpoint asymptotic for every maximal field	proved	explicit Green-form reconstruction
No-log realization equals the Friedrichs extension	proved	canonical closed-form identification
Quantum stress, gravitational backreaction, and geometric singularity resolution	outside present theorem	separate downstream problem

D Formal development and collaboration

The theorem package presented here is part of a larger Lean 4 research development. The paper-scoped theorem map above records the authority surface used in the manuscript and keeps repository-specific implementation detail concentrated in the appendices.

The present manuscript is restricted to the Schwarzschild radial scalar operator, its terminal self-adjoint extension theory, the maximal endpoint trace including its derivative asymptotic, and the finite full-potential ground-state/Friedrichs selection theorem. All downstream quantum-field and gravitational developments remain logically separate from this theorem package.

The formal source corresponding to the frozen revision is available for technical review upon request.

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