Endpoint Rigidity in Periodic Navier–Stokes Flow

Terminal magnification, intrinsic restart, and analytic rigidity at the $L_t^\infty L_x^3$ boundary

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Abstract

Let

$$u : [0, T_{\max}) \to H^2(\mathbb{T}^3)$$

be the canonical smooth forced periodic three-dimensional Navier–Stokes trajectory. Papers I and II establish the finite-maximal continuation anatomy below the velocity endpoint, including bounded kinetic energy, eventual H^2 divergence, escape of H^2 mass beyond every fixed Fourier cutoff, terminal nonlinear derivative-weighted work, collapse of uniform continuation reserve, and failure of every nonendpoint Prodi–Serrin control established in Paper II. The remaining critical boundary is $L_t^\infty L_x^3$.

We prove

$$\boxed{\sup_{0<t<T_{\max}} \|u(t)\|_{L^3(\mathbb{T}^3)} < \infty \quad \Longrightarrow \quad T_{\max} = \infty.} \tag{1}$$

Equivalently,

$$\boxed{T_{\max} < \infty \quad \Longrightarrow \quad \limsup_{t \uparrow T_{\max}} \|u(t)\|_{L^3(\mathbb{T}^3)} = \infty.} \tag{2}$$

The conclusion is terminal infinite limsup; the theorem is stated at exactly that temporal strength. The analytical endpoint principle is classical in the work of Escauriaza–Seregin–Šverák. The contribution here is a machine-checked realization for the exact forced periodic maximal trajectory, including finite-scale forcing, blow-up-limit autonomy, retained-state compactness, intrinsic restart, and analytic rigidity.

Under the contradictory assumptions $T_{\max} < \infty$ and bounded terminal L^3, parabolic magnification at an endpoint singular point yields a nontrivial unforced suitable ancient state $U \in L_t^\infty L_x^3$. The limiting state has canonical exterior vorticity vanishing on every negative terminal window. Every negative interval contains an intrinsic global $H^1 \cap L^3$ restart slice, and weak–strong identification shows that the resulting mild restart is the same ancient state. Positive-time spatial analyticity is classical; the proof obligation here is to identify the literal Oseen history and its analytic representative with that same retained state. Exterior curl zero then propagates across each connected spatial slice, harmonic L^3 rigidity produces zero slices in every negative interval, and weak critical time continuity forces $U \equiv 0$, contradicting retained terminal nontriviality.

Several exact obstruction results record the admissible boundaries of the argument. A separate companion theorem quantifies critical-root contraction under pre-existing smallness and is logically independent of the endpoint contradiction.

1 Introduction

Papers I and II establish one finite-maximal periodic Navier–Stokes trajectory through two complementary layers. Paper I constructs the strong periodic evolution on the canonical maximal interval and proves bounded kinetic energy together with eventual H^2 divergence, high-frequency escape, fixed positive heat-scale regularity, and collapse of uniform continuation reserve. Paper II places on that same trajectory the exact finite-annulus transfer law, terminally unbounded nonlinear H^2 work beyond every prescribed spectral floor, nonintegrable physical gradient and vorticity suprema, and failure of every nonendpoint Prodi–Serrin class treated in Paper II.[1, 2]

The present paper completes the $L_t^\infty L_x^3$ boundary in the continuation anatomy developed in Papers I and II. The analytical endpoint principle has classical lineage in Escauriaza–Seregin–Šverák [5]; the theorem proved here is its realization inside the exact machine-checked forced periodic continuation program of Papers I and II, with an established terminal singularity, asymptotic force disappearance, retained ancient-state identity, intrinsic restart, and parabolic analytic rigidity.

A recent forced finite-time blowup construction provides a contemporary example of singular behavior prepared by smooth external forcing [8]. For the present analysis, the relevant structural point is the scaling law: terminal parabolic magnification retains the force at every finite scale and sends it to zero on fixed compact blow-up carriers. The resulting limit is therefore autonomous, and the reachability problem for such autonomous terminal cores becomes a separate question of dynamical realization.

The paper records obstruction results as mathematical boundary statements. Each one identifies the precise scope of a candidate inference, geometry, or coercive estimate and thereby fixes the admissible structure used by the next construction. The endpoint contradiction itself runs from scale-critical nonendpoint failure through terminal magnification, ancient-state persistence, exterior recovery, intrinsic restart, authoritative Oseen analyticity, spatial rigidity, and endpoint continuation. The critical-root contraction theorem is recorded on a separate companion branch with its declared small-root hypothesis.

We consider

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u + f, \qquad \nabla \cdot u = 0, \qquad \nu > 0, \tag{3}$$

on the unit three-torus $\mathbb{T}^3$, with smooth real divergence-free initial data and smooth prescribed periodic forcing.

1.1 The terminal state inherited from Papers I and II

The preceding papers prove the following complete simultaneous anatomy [1, 2]. If $T_{\max} < \infty$, then

$$\boxed{T_{\max} < \infty \Longrightarrow \begin{cases} \sup_{t<T_{\max}} \frac{1}{2}\|u(t)\|_2^2 < \infty, \\ \|u(t)\|_{H^2} \to \infty, \\ \forall K < \infty : \quad \mathsf{H}_{2,>K}(u(t)) \to \infty, \\ \forall a > 0 : \quad \sup_{t<T_{\max}} \mathsf{H}_2(e^{a\Delta}u(t)) < \infty, \\ \forall s < T_{\max},\ \forall K_0,\ \forall M \in \mathbb{R} : \quad \exists K_0 < K < L,\ \exists t \in (s, T_{\max}) : \\ \qquad\qquad M < \mathsf{W}^{\mathrm{NL}}_{K<L}(s,t), \\ \|\nabla u\|_\infty \notin L^1_t, \qquad \|\operatorname{curl} u\|_\infty \notin L^1_t, \\ \|u\|_\infty^2 \notin L^1_t, \\ \forall q > 3 : \quad \|u\|_q^{2q/(q-3)} \notin L^1_t, \\ \text{uniform positive bounded-control continuation reserve collapses.} \end{cases}} \tag{4}$$

The display is organizational. These statements are simultaneous necessary consequences of finite maximality and are used without imposing a causal ordering among spectral escape, terminal nonlinear work, deformation, vorticity growth, Prodi–Serrin failure, and continuation-reserve collapse. [1]

1.2 The endpoint bottleneck

For $q > 3$, set

$$p(q) = \frac{2q}{q-3}, \qquad \frac{2}{p(q)} + \frac{3}{q} = 1. \tag{5}$$

Under parabolic scaling

$$u_r(x,t) = r\,u(rx, r^2t),$$

one has

$$\|u_r(t)\|_{L^q(\mathbb{R}^3)} = r^{1-3/q}\|u(r^2t)\|_{L^q(\mathbb{R}^3)}. \tag{6}$$

At $q = 3$,

$$1 - \frac{3}{3} = 0, \qquad p(q) \uparrow \infty \quad (q \downarrow 3), \qquad \|u_r(t)\|_3 = \|u(r^2t)\|_3. \tag{7}$$

Thus

$$\boxed{\text{the } L^\infty_t L^3_x \text{ endpoint requires its own scale-critical compactness–rigidity argument.}} \tag{8}$$

The endpoint is scale invariant, and its continuation theory is therefore carried by a compactness–rigidity mechanism.

The $L^\infty_t L^3_x$ regularity principle has classical analytical lineage in the theorem of Escauriaza, Seregin and Šverák [5]. The contribution of the present paper is the endpoint realization within the exact

[1] *Formal counterpart:* `Serrin/Canonical.lean`.

forced periodic continuation framework of Papers I and II: the actual canonical physical L^3 norm, an established endpoint singular point, finite-scale forcing with local C^m disappearance under blow-up, retained-state ancient extraction, intrinsic restart and same-state identification, and the parabolic analytic restart rigidity used below.

Theorem 1.1 (Periodic endpoint continuation)**.** *For the canonical smooth forced periodic evolution,*

$$\boxed{\sup_{0<t<T_{\max}} \|u(t)\|_{L^3(\mathbb{T}^3)} < \infty \quad \Longrightarrow \quad T_{\max} = \infty.} \tag{9}$$

[2]

Finite maximality therefore implies unbounded L^3 on the maximal interval, and smoothness on every compact subinterval localizes that conclusion to every terminal tail.

Theorem 1.2 (Terminal L^3 unboundedness)**.** *If $T_{\max} < \infty$, then*

$$\forall s < T_{\max},\ \forall M \in \mathbb{R},\ \exists t \in (s, T_{\max}) : \qquad M < \|u(t)\|_{L^3(\mathbb{T}^3)}. \tag{10}$$

Equivalently,

$$\boxed{\limsup_{t\uparrow T_{\max}} \|u(t)\|_{L^3(\mathbb{T}^3)} = \infty.} \tag{11}$$

[3]

The proof of Theorem 1.1 follows one physical state through changes of scale, carrier, continuation, and spatial representation:

$$\boxed{\begin{aligned} T_{\max} < \infty, \quad M_3 < \infty &\Longrightarrow \text{endpoint singular point and critical concentration} \\ &\Longrightarrow \text{nontrivial unforced ancient state } U \\ &\Longrightarrow \text{exterior canonical curl zero} \\ &\Longrightarrow \text{intrinsic restart } V = U^{[t_0]} \\ &\Longrightarrow \text{analyticity of the same ancient state} \\ &\Longrightarrow \text{curl-zero slab in every negative interval} \\ &\Longrightarrow U = 0 \\ &\Longrightarrow \bot. \end{aligned}} \tag{12}$$

The next sections make each representation change explicit.

The dependency structure consists of the headline endpoint chain together with a separate companion

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[2] *Formal counterpart:* `Tier30.Research.NavierStokes.EndpointRigidity.periodic_boundedL3_implies_continuation`.

[3] *Formal counterpart:* `Tier30.Research.NavierStokes.EndpointRigidity.finiteBreakdown_L3_terminallyUnbounded`.

refinement branch. The headline endpoint proof is

$$\boxed{\begin{array}{c}\text{Paper-II finite-maximal anatomy}\\ \downarrow\\ q=3\text{ scale-critical bottleneck}\\ \downarrow\\ \text{terminal singular point and critical concentration}\\ \downarrow\\ \text{nontrivial autonomous ancient state}\\ \downarrow\\ \text{exterior canonical curl zero}\\ \downarrow\\ \text{intrinsic restart and same-state identification}\\ \downarrow\\ \text{authoritative Oseen analyticity}\\ \downarrow\\ \text{whole-slice curl rigidity}\\ \downarrow\\ \text{terminal physical } L^3 \text{ unboundedness.}\end{array}} \tag{13}$$

The companion refinement theorem occupies the separate branch:

$$\boxed{\text{pre-existing small critical root} \Longrightarrow \text{one-step contraction} \Longrightarrow \text{all-radius positive-power decay.}} \tag{14}$$

The obstruction decisions of Section 4 specify the hypotheses carried by each arrow in (13); the scale-coherence theorem (14) remains on the companion branch.

2 Terminal scaling and critical concentration

The physical and internal clocks satisfy

$$\tau = 2\pi t, \qquad \nu_{\text{internal}} = 2\pi\nu_{\text{physical}}. \tag{15}$$

Finite maximality yields an actual endpoint singular point. Thus there exists $x_* \in \mathbb{T}^3$ such that, with $z_* = (x_*, T_{\max})$, the terminal point is singular for the canonical physical maximal history. [4]

For $r > 0$, set

$$u_r(y,s) = r\,u(x_* + ry, T_{\max} + r^2 s), \tag{16}$$
$$p_r(y,s) = r^2 p(x_* + ry, T_{\max} + r^2 s), \tag{17}$$
$$f_r(y,s) = r^3 f(x_* + ry, T_{\max} + r^2 s). \tag{18}$$

Then

$$\partial_s u_r + (u_r\cdot\nabla)u_r + \nabla p_r = \nu\Delta u_r + f_r, \qquad \nabla\cdot u_r = 0. \tag{19}$$

[4] *Formal counterpart:* `EndpointRigidityMechanism.lean`.

2.1 Spectral escape and terminal magnification

Papers II and III use different representations of the same finite-maximal trajectory:

Paper II: global spectral view	**Paper III: local parabolic view**
$\lvert k\rvert \to \infty$	$r \downarrow 0$
$\mathsf{H}_{2,>K}(u) \to \infty$	(u_r, p_r) viewed at fixed unit scale
finite-annulus nonlinear H^2 work	$C(r) + D(r)$ critical concentration
continuation-reserve collapse	terminal-core admissibility and restart

(20)

High-frequency escape and physical-space concentration are treated as distinct proved representations of the same proposed finite-breakdown history, with no pointwise or scale-by-scale identification imposed between them. The carrier relation is

$$\boxed{u \longrightarrow u_{r_n} \longrightarrow U,} \tag{21}$$

and the task is to determine which identities and response laws survive those changes of carrier.

2.2 Asymptotic autonomy

For every compact $K \Subset \mathbb{R}^3 \times (-\infty, 0)$ and every $m \in \mathbb{N}$,

$$\|f_r\|_{C^m(K)} \longrightarrow 0 \qquad (r \downarrow 0). \tag{22}$$

[5] Thus the force remains exact at every finite scale and disappears in the limiting local carrier. Any sufficiently strong terminal limit satisfies

$$\partial_s U + (U \cdot \nabla)U + \nabla P = \nu \Delta U, \qquad \nabla \cdot U = 0. \tag{23}$$

Consequently,

$$\boxed{\text{the terminal core is governed by the autonomous limiting equation generated by parabolic magnification.}} \tag{24}$$

This is the scaling structure of the blow-up passage: the physical history is forced, and the retained terminal core is asymptotically autonomous.

2.3 Critical local quantities and retained terminal concentration

For $z_0 = (x_0, t_0)$ and $r > 0$, define

$$Q_r(z_0) := B_r(x_0) \times (t_0 - r^2, t_0), \tag{25}$$

$$C(z_0, r; u) := r^{-2} \int_{Q_r(z_0)} |u|^3, \tag{26}$$

$$D(z_0, r; p) := r^{-2} \int_{Q_r(z_0)} |p - (p)_{B_r}(t)|^{3/2}. \tag{27}$$

Both C and D are invariant under terminal rescaling.

[5] *Formal counterpart:* `EndpointRigidityMechanism.lean`.

Assume for contradiction

$$T_{\max} < \infty, \qquad M_3 := \sup_{0<t<T_{\max}} \|u(t)\|_{L^3} < \infty. \tag{28}$$

Finite maximality supplies $\varepsilon_* > 0$ and $r_n \downarrow 0$ such that, for every fixed $R > 0$,

$$C((0,0), R; u_{r_n}) + D((0,0), R; \widetilde{p}_{r_n}) \geq c_* \varepsilon_* \tag{29}$$

eventually in n, with $\widetilde{p}_{r_n}$ the canonical ball-gauge pressure.

To retain positive mass on a fixed ancient carrier away from the terminal face, set

$$S_N := \overline{B}_1(0) \times \left(-\frac{1}{N+1}, 0\right), \qquad K_N := \overline{B}_{N+1}(0) \times \left[-(N+1), -\frac{1}{N+1}\right]. \tag{30}$$

The endpoint slice bounds imply

$$\sup_{n \geq n_0(N)} \left(\int_{S_N} |u_{r_n}|^3 + \int_{S_N} |\widetilde{p}_{r_n}|^{3/2} \right) \longrightarrow 0 \qquad (N \to \infty). \tag{31}$$

Hence some compact ancient carrier retains positive combined mass. The terminal-profile theorem then separates velocity from pressure and yields

$$\boxed{\liminf_{n \to \infty} \|u_{r_n}\|_{L^3(K_{N_0})} \geq c_0 > 0.} \tag{32}$$

[6]

Proposition 2.1 (Velocity-only terminal profile)**.** *Under (28), an endpoint singular point admits a terminal rescaling sequence and a fixed compact ancient carrier K_{N_0} for which (32) holds. The lower bound is a velocity-only conclusion and requires no pressure convergence hypothesis.*

The pressure separation uses an explicit decay estimate. On the fixed normalized time window I used by the one-cylinder contradiction, and for the normalized small-amplitude family on that window, write $\widehat{p}_n$ for the normalized pressure and let e_n denote the harmonic-remainder error. There are finite constants G, B and a fixed core radius $r_0 > 0$ such that $\|e_n\| \to 0$ and, for $0 < r < r_0$, eventually in n,

$$\boxed{\widehat{D}_n(r) \leq 2^{1/2} \left[G r^{5/2} (2 + B)^{3/2} + r^{-2} (2\|e_n\|)^{3/2} \right],} \tag{33}$$

where

$$\widehat{D}_n(r) := r^{-2} \int_{I \times B_r} \left| \widehat{p}_n(x,t) - \frac{1}{|B_r|} \int_{B_r} \widehat{p}_n(y,t)\, dy \right|^{3/2} \mathrm{d}x\, \mathrm{d}t. \tag{34}$$

The canonical ball gauge used by the critical quantity obeys, on the same cylinder,

$$\boxed{D(r; \mathrm{normalize}_{B_r}\, p) \leq 2^{3/2} D(r; p).} \tag{35}$$

Thus the pressure oscillation is made small by first selecting r so that the $r^{5/2}$ term in (33) is small and then taking n large so that the error term is small. Together with the cubic velocity budget, the vanishing rescaled force, and the epsilon-regularity threshold, this yields a positive lower bound for the velocity cubic mass along every singular sequence. This is the pressure mechanism behind Proposition 2.1; the terminal-strip estimate then moves the positive velocity mass from the terminal cylinder to the fixed ancient carrier K_{N_0}. [7]

[6] *Formal counterpart:* `MaximalTerminalCriticalProfile.lean;MaximalTerminalNontriviality.lean`.

[7] *Formal counterpart:* `ForceAwareOriginalPressureDecay.lean;PhysicalSmoothPressureBallGauge.lean;MaximalTerminalCriticalProfile.lean`.

3 Companion critical-scale coherence under refinement

The endpoint is scale invariant. Refinement is therefore organized through a dynamical contraction law for the canonical critical root. Write schematically

$$\mathcal{R}(z,r) := C(z,r;u)^{1/3} + D(z,r;p)^{2/3}, \tag{36}$$

with the canonical normalized pressure and source geometry used throughout the local theory.

For every positive viscosity and every prescribed contraction $0<\kappa<1$, the scale-uniform one-step theorem selects a fraction $0<\theta<1/4$, a positive source-root threshold, and a positive maximum physical scale such that sufficiently small original forced profiles satisfy

$$\boxed{\mathcal{R}(z,8\theta r) \le \kappa\big(\mathcal{R}(z,8r)+\mathcal{F}(z,r)\big),} \tag{37}$$

where $\mathcal{F}(z,r)$ is the actual enlarged-carrier force floor. Smooth forcing satisfies

$$\mathcal{F}(z,r) \longrightarrow 0 \qquad (r\downarrow 0). \tag{38}$$

Geometric iteration then gives positive real-power decay at every smaller radius. In particular, there are $\alpha>0$ and finite geometry-dependent constants such that, under the same source-root hypothesis,

$$\mathcal{R}(z,\rho) \lesssim \left(\frac{\rho}{r}\right)^{\alpha}\big(\mathcal{R}(z,8r)+\text{force budget at scale } r\big), \qquad 0<\rho\le 8r. \tag{39}$$

[8]

Equations (37)–(39) give the title phrase *critical scale coherence* a precise meaning:

$$\boxed{\text{sufficiently small critical root control} \Longrightarrow \text{quantitative coherence under refinement.}} \tag{40}$$

The theorem assumes the declared source-root smallness and yields the stated contraction and power decay within that regime. Equations (37)–(39) therefore remain a companion refinement result, logically independent of Theorem 1.1. The result is retained as a companion theorem because it identifies the contracting local regime available once small critical-root control is already present, and because it gives a precise refinement counterpart to Paper II's fine-scale spectral escape.

4 Endpoint obstruction decisions

The endpoint development contains several exact obstruction results. Each fixes a mathematical boundary used by the proof and is presented as a theorem-level scope decision.

[8] *Formal counterpart:* `ScaleUniformPhysicalOneStep.lean;QuantitativeOriginalPhysicalIteration.lean;AllRadiusPhysicalRootPowerDecay.lean`.

4.1 Velocity nontriviality requires a velocity-only terminal profile

A combined lower bound of the form

$$\int_K |u_n|^3 + \int_K |p_n|^{3/2} \geq c > 0 \tag{41}$$

controls the total critical concentration. The velocity share is then identified separately. The theorem therefore establishes the velocity-only estimate

$$\boxed{\int_K |u_n|^3 > 0} \tag{42}$$

through pressure separation and terminal-strip tightness before strong local L^3 convergence is used to obtain $U \neq 0$. [9]

4.2 Finite-scale forcing persists under rescaling

For every positive blow-up scale r, the rescaled trajectory solves

$$\partial_s u_r + (u_r \cdot \nabla) u_r + \nabla p_r = \nu \Delta u_r + f_r, \qquad f_r(y,s) = r^3 f(x_* + ry, T_{\max} + r^2 s). \tag{43}$$

At every finite r, the rescaled equation retains f_r. The asymptotic local statement is:

$$\boxed{\forall m,\ K \Subset \mathbb{R}^3 \times (-\infty, 0): \qquad \|f_r\|_{C^m(K)} \to 0 \quad (r \downarrow 0).} \tag{44}$$

Thus autonomy is a property of the limiting core produced by the exact finite-scale forced dynamics. [10]

4.3 The ancient laws are formulated on their exact domain

The compactness passage produces the limiting equations on

$$\Omega_- := \mathbb{R}^3 \times (-\infty, 0), \tag{45}$$

through the domain-aware suitable carrier. Any extension of the weak equations, local-energy inequality, pressure, or gradient beyond that domain. The domain principle is

$$\boxed{\text{law on the actual domain} \neq \text{law silently extended beyond the domain.}} \tag{46}$$

Accordingly the ancient solution is first established as a suitable weak solution on Ω_-; every later time translation, zero extension, or restart domain is obtained by a separate theorem. [11]

[9] *Formal counterpart:* `MaximalTerminalCriticalProfile.lean`;`MaximalTerminalNontriviality.lean`.

[10] *Formal counterpart:* `BlowupSequence.lean`;`EndpointRigidityMechanism.lean`;`AncientLimit.lean`.

[11] *Formal counterpart:* `DomainSuitableWeak.lean`;`AncientLimit.lean`.

4.4 Weak terminal momentum trace leaves the velocity contribution undetermined in strong exterior terminal trace

The ancient extraction yields compact-tested momentum convergence

$$\int_{\mathbb{R}^3} U(x,t) \cdot \phi(x)\, dx \longrightarrow 0 \qquad (t \uparrow 0) \tag{47}$$

for compact smooth tests. This leaves the velocity contribution undetermined in a strong cubic trace, pointwise exterior trace, or the uniform exterior zero trace needed to remove the terminal commutator. Those stronger statements are proved separately for theorem-owned continuous exterior representatives:

$$\boxed{\sup_{|x|>R,\, -\delta<t<0} \left(|\widetilde{U}(x,t)| + |\widetilde{G}(x,t)| + |\widetilde{\omega}(x,t)|\right) \longrightarrow 0 \quad (\delta \downarrow 0).} \tag{48}$$

The zero-extension argument uses this strong exterior trace as its boundary input. [12]

4.5 Fixed-query recovery requires a query-retaining geometry

For the centered moving-hole Carleman family, the normalized inner radius grows, and the weight at every fixed physical query outside that hole satisfies

$$w_n(x_{\mathrm{q}}, t_*) \longrightarrow 0. \tag{49}$$

The formal obstruction is stronger: after some stage the weight is strictly below one,

$$\exists N\ \forall n \geq N : \qquad w_n(x_{\mathrm{q}}, t_*) < 1, \tag{50}$$

so the eventual lower bound required for fixed-query recovery is unavailable. The centered family therefore supplies the decay estimate. Fixed-query recovery is carried by a query-retaining geometry. [13]

4.6 Annular recovery requires compatible time orientation

One future-shifted annular lane requires the inner-face decay orientation

$$t_{\mathrm{earlier}} < t_{\mathrm{shift}}, \tag{51}$$

and the ancient recovery core requires

$$t_{\mathrm{shift}} < t_{\mathrm{rec}} < t_{\mathrm{earlier}}. \tag{52}$$

Equations (51)–(52) require distinct orderings. The admissible recovery architecture therefore uses the shift-free spatial formulation with a calibrated preterminal core. [14]

[12] *Formal counterpart:* `MaximalAncientZeroTerminalTesting.lean;MaximalAncientExteriorStrongZeroNativeVorticity.lean`.

[13] *Formal counterpart:* `ActualVanishingScaleRecoveryObstruction.lean`.

[14] *Formal counterpart:* `ActualFutureShiftedAffineTimeOrientation.lean;ActualExteriorAnnularESSPreterminalCoreRecovery.lean`.

4.7 Past-decaying temporal weight is paired with spatial pseudoconvex coercivity

A backward-time exponential chosen to decay into the ancient past removes the unwanted temporal growth and produces a negative Dirichlet coefficient in the corresponding whole-spacetime Carleman estimate. The repaired coercive combination is

$$\boxed{\text{past-decaying temporal estimate} + \frac{1}{4}\,\text{spatial pseudoconvex estimate} \Longrightarrow c_D^{\text{comb}} > 0, \quad c_M^{\text{comb}} > 0.} \tag{53}$$

The temporal and spatial weights therefore carry complementary sign obligations in the combined coercive estimate. [15]

4.8 Regularity transfer is carried by same-state identity

A strong restart V from the slice $U(t_0)$ is initially a comparison solution. Transfer of its regularity to the retained ancient state is mediated by the identity

$$V \in C^\omega, \qquad V(0) = U(t_0), \qquad V = U^{[t_0]} \Longrightarrow U^{[t_0]} \in C^\omega. \tag{54}$$

The required identification is the weak–strong identity

$$\boxed{V = U^{[t_0]}} \tag{55}$$

on a positive plateau. Every downstream regularity statement is transferred only after this identity is established.

4.9 Inhomogeneous heat smoothing retains the zero Fourier mode

For the inhomogeneous Sobolev tower, the all-mode estimate

$$\|e^{t\Delta}u\|_{H^{s+1}} \lesssim t^{-1/2}\|u\|_{H^s} \tag{56}$$

requires a zero-mode correction for uniform positive times, because the zero Fourier mode is unchanged by the heat multiplier and carries no derivative gain. The correct inhomogeneous estimate retains the zero-mode contribution,

$$\boxed{\|e^{\nu t\Delta}u\|_{H^{s+1}} \leq \sqrt{1 + (2\nu t)^{-1}}\,\|u\|_{H^s}.} \tag{57}$$

A pure $t^{-1/2}$ gain is recovered on the homogeneous mean-zero component. This distinction fixes which part of the heat semigroup is actually responsible for derivative recovery in the restart bootstrap. [16]

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[15] *Formal counterpart:* `ExteriorQuadraticPastCarleman.lean;ExteriorPseudoconvexVorticity.lean;ActualOriginalQuadraticPastAbsorption.lean`.

[16] *Formal counterpart:* `SmoothBootstrap/HsHeat.lean;SmoothBootstrap/HsPropagation.lean`.

4.10 The analytic radius follows the parabolic restart scale

The restart begins from $H^1 \cap L^3$ data at elapsed time zero. The analytic carrier therefore uses a radius that vanishes at the entrance:

$$\boxed{R_\sigma(t) = \sigma\sqrt{t}, \qquad R_\sigma(t) \downarrow 0 \ (t \downarrow 0).} \tag{58}$$

This parabolic entrance geometry matches the parabolic source/kernel clocks used in the Oseen history. [17]

4.11 Obstruction decision map

Analytic question or candidate lane	Exact boundary	Structure used in the final proof
Combined $C + D$ concentration	Pressure may carry the combined lower bound	Velocity-only terminal L^3 lower bound
Finite-scale force scaling	f_r remains in every finite rescaled equation	Local C^m disappearance in the blow-up limit
Ancient-carrier domain	Compactness owns the laws on Ω_-	Domain-aware suitable carrier and explicit domain transport
Weak trace $\Rightarrow$ strong trace	Compact-tested terminal momentum leaves the velocity contribution undetermined in exterior strong zero trace	Separate continuous exterior strong-trace theorem
Centered vanishing-scale recovery	Fixed-query weight tends to zero; required lower bound fails	Query-retaining exterior recovery geometry
Future-shifted annular lane	Decay and recovery time orders are incompatible	Shift-free spatial recovery / calibrated preterminal core
Past-decaying temporal weight	Dirichlet coefficient requires spatial coercive compensation	Composite temporal + spatial pseudoconvex coercivity
Smooth comparison restart	Regularity transfer requires state identification	Weak–strong equality $V = U^{[t_0]}$
All-mode $t^{-1/2}$ heat smoothing	Zero Fourier mode carries no derivative gain	Inhomogeneous factor $\sqrt{1+(2\nu t)^{-1}}$; homogeneous mean-zero gain
Analytic entrance radius	Rough restart trace selects a vanishing parabolic entrance budget	Parabolic radius $R_\sigma(t) = \sigma\sqrt{t}$

Table 1: Endpoint obstruction decisions retained as mathematical results.

The obstruction decisions and constructive theorems therefore occupy one proof architecture: each boundary statement identifies the admissible hypothesis used by the next theorem in the endpoint contradiction.

[17] *Formal counterpart:* `RestartTemporalOseenParabolicTaylorMass.lean`.

5 Ancient suitable limit and exterior vorticity

Set

$$\Omega_- := \mathbb{R}^3 \times (-\infty, 0). \tag{59}$$

A nested diagonal subsequence $r_{n_k} \downarrow 0$ yields fields

$$U : \Omega_- \to \mathbb{R}^3, \qquad G : \Omega_- \to \mathbb{R}^{3\times 3}, \qquad P : \Omega_- \to \mathbb{R}$$

with

$$u_{r_{n_k}} \to U \qquad \text{strongly in } L^3_{\text{loc}}(\Omega_-), \tag{60}$$

$$\nabla u_{r_{n_k}} \rightharpoonup G \qquad \text{on compact ancient carriers,} \tag{61}$$

$$G = \nabla U \qquad \text{in } \mathcal{D}'(\Omega_-), \tag{62}$$

$$\partial_t U + (U \cdot \nabla)U + \nabla P = \nu \Delta U \qquad \text{in } \mathcal{D}'(\Omega_-), \tag{63}$$

$$\nabla \cdot U = 0 \qquad \text{in } \mathcal{D}'(\Omega_-). \tag{64}$$

The local-energy inequality holds on the same domain, and

$$U \in L^\infty((-\infty, 0); L^3(\mathbb{R}^3)). \tag{65}$$

For every compact smooth vector test ϕ,

$$\int_{\mathbb{R}^3} U(x,t) \cdot \phi(x)\, dx \longrightarrow 0 \qquad (t \uparrow 0). \tag{66}$$

[18]

The velocity-only lower bound (32) and strong local L^3 convergence give

$$\exists R > 0 : \qquad \int_{Q_R^-} |U|^3 > 0. \tag{67}$$

We write AncientNontrivial(U) for (67). Thus completion changes the carrier without erasing the state:

$$\boxed{\text{physical trajectory} \longrightarrow \text{terminal rescalings} \longrightarrow \text{ancient carrier with } U \neq 0.} \tag{68}$$

Let

$$\omega := \nabla \times U \tag{69}$$

be the canonical curl induced by the retained gradient G.

Theorem 5.1 (Exterior strong terminal zero trace)**.** *For every lower time $\ell < 0$, there exist a finite exterior radius R_ℓ and continuous exterior representatives*

$$\widetilde{U}, \qquad \widetilde{G}, \qquad \widetilde{\omega} := \nabla \times \widetilde{U} \tag{70}$$

on the closed terminal exterior carrier

$$\mathcal{E}_{\ell, R_\ell} := \{(x,t) : |x| > R_\ell,\ \ell < t \leq 0\} \tag{71}$$

[18] *Formal counterpart:* `MaximalEndpointIntrinsicNontriviality.lean`.

which agree almost everywhere for $t<0$ with U, G, and the canonical curl of G, and satisfy

$$\widetilde{U}(x,0)=0, \qquad \widetilde{G}(x,0)=0, \qquad \widetilde{\omega}(x,0)=0. \tag{72}$$

Moreover the trace is spatially uniform on the exterior:

$$\boxed{\lim_{\delta\downarrow 0}\ \sup_{\substack{|x|>R_\ell\\ -\delta<t<0}} \left(|\widetilde{U}(x,t)|+|\widetilde{G}(x,t)|+|\widetilde{\omega}(x,t)|\right)=0.} \tag{73}$$

This is the terminal boundary statement used by the exterior Carleman argument. It strengthens the weak velocity trace in Eq. (66) through a separately established exterior regularity theorem. The exterior regularity construction first selects uniformly continuous representatives of the same ancient velocity and gradient on a closed exterior terminal window. Their theorem-owned terminal pair vanishes at $t=0$; uniform continuity then gives the spatially uniform convergence in Eq. (73), and the fixed linear curl map transfers the same conclusion to $\widetilde{\omega}$. In particular every compact exterior test sees zero terminal vorticity trace, which is the boundary input needed to remove the shrinking terminal commutator below. [19]

Lemma 5.2 (Zero extension across the terminal face)**.** *Fix the exterior representatives supplied by Theorem 5.1 and extend them by zero for $t\geq 0$. The extended vorticity satisfies the same distributional transport–heat law across $t=0$; no terminal delta source is created.*

The open zero region is obtained from the strong exterior trace in Theorem 5.1, which supplies the boundary datum for the zero extension. If $\bar{U},\bar{\omega}$ denote the zero-extended exterior representatives, then for every compact spacetime test φ one obtains

$$\boxed{\int \bar{\omega}\cdot(-\partial_t\varphi-\nu\Delta\varphi)+\int(\bar{U}\times\bar{\omega})\cdot(\nabla\times\varphi)=0.} \tag{74}$$

The proof uses negative-time test exhaustion and a shrinking terminal cutoff. The only boundary contribution is the terminal commutator

$$\int \bar{\omega}(x,t)\cdot\varphi(x,t)\,\eta_n'(t)\ \mathrm{d}x\,\mathrm{d}t, \tag{75}$$

which tends to zero by the compact zero-trace theorem. Hence the extension is distributionally valid through the terminal clock. By construction,

$$\bar{\omega}(x,t)=0 \qquad (t>0) \tag{76}$$

throughout the exterior domain, providing an actual open *future* zero region. [20]

Lemma 5.3 (First negative-time exterior zero window)**.** *There exist a finite radius $R_*>0$ and a duration $\delta_*>0$ such that the same zero-extended vorticity satisfies*

$$\boxed{\bar{\omega}(x,t)=0 \qquad (|x|>R_*,\ -\delta_*<t<0).} \tag{77}$$

[19] *Formal counterpart:* `MaximalAncientExteriorStrongZeroNativeVorticity.lean`.
[20] *Formal counterpart:* `ActualFutureZeroNativeVorticityWeakHeat.lean;MaximalAncientExteriorZeroExtendedVorticityHeat.lean`.

Equation (77) is the output of the balanced moving-center Carleman recovery built on the established strong exterior trace. Starting from the open future zero region (76), the exterior-annular Carleman estimate is applied to the same vorticity field. The temporal weight decays into the ancient past and has a negative Dirichlet contribution; adding the spatial pseudoconvex estimate restores positive Dirichlet and moment coefficients. Gaussian tails, the local gradient-energy bounds, and the explicit polynomial heat-jet estimates remove the cutoff-gradient and heat-cutoff errors. The moving-center normalization retains the physical recovery query and suppresses the error faces, yielding (77). [21]

Theorem 5.4 (Exterior vorticity vanishing). *For every $T < 0$, there exists $R_T < \infty$ such that*

$$\boxed{\omega = 0 \quad a.e.\ on \quad \{(x,t) : |x| > R_T,\ T < t < 0\}.} \tag{78}$$

The field in (78) is the curl of the same spatial gradient G appearing in (64).

The proof now begins from the genuine negative-time window in Lemma 5.3. Resetting the moving-center clock inside a known exterior zero window gives an earlier exterior zero window for the same field. Repetition covers any prescribed interval $(T, 0)$. At every stage the mollified/classical representatives are identified almost everywhere with the curl of the stored gradient G, so the final conclusion is stated for the single canonical ancient vorticity retained throughout the construction. [22]

The exterior theorem organizes the terminal geometry, and the full dynamics retain their global coupling. Pressure remains nonlocal, the quadratic response couples spatial regions and scales, and the subsequent restart uses global heat/Leray evolution. What is retained is identity of the state through these contextual couplings:

$$\boxed{\text{local core identifiable within a globally coupled evolution.}} \tag{79}$$

6 Intrinsic restart and weak–strong identification

Proposition 6.1 (Global $H^1 \cap L^3$ restart slice). *For every $a < b < 0$, there exists $t_0 \in (a, b)$ such that*

$$U(t_0) \in H^1(\mathbb{R}^3) \cap L^3(\mathbb{R}^3), \qquad \nabla \cdot U(t_0) = 0. \tag{80}$$

The H^1 membership concerns the same literal ancient velocity and the same retained gradient used in the compactness construction.

The global conclusion is stronger than the local H^1 information supplied directly by suitability, and the proof uses the exterior vorticity theorem essentially. Choose an exhaustive good time $t_0 \in (a, b)$ for which the native curl

$$\omega_0 := \nabla \times U(t_0) \tag{81}$$

belongs to

$$\boxed{\omega_0 \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3), \qquad \omega_0 = 0 \text{ a.e. on } \{|x| > R\}} \tag{82}$$

[21] *Formal counterpart:* `ExteriorQuadraticPastCarleman.lean;ActualOriginalQuadraticPastAbsorption.lean; MaximalAncientExteriorTerminalWindow.lean`.

[22] *Formal counterpart:* `MaximalAncientExteriorFutureQueryRecovery.lean;MaximalAncientCanonicalExterior VorticityVanishing.lean`.

for some finite R. Thus the closed-ball truncation of ω_0 agrees almost everywhere with ω_0 and is compactly supported.

Let $\mathcal{B}[\omega_0]$ be the whole-space Biot–Savart reconstruction of this compactly supported vorticity. The established mapping bounds give

$$\boxed{\mathcal{B}[\omega_0] \in L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3),} \tag{83}$$

and the reconstruction has the same distributional vorticity–Poisson source as the literal slice $U(t_0)$. Since $U(t_0) \in L^3$ already, the difference

$$H := U(t_0) - \mathcal{B}[\omega_0] \tag{84}$$

is distributionally harmonic and belongs to $L^3(\mathbb{R}^3)$. Harmonic L^3 rigidity gives $H = 0$, hence

$$\boxed{U(t_0) = \mathcal{B}[\omega_0] \quad \text{a.e.}, \qquad U(t_0) \in L^2(\mathbb{R}^3).} \tag{85}$$

It remains to globalize the gradient. The weak curl and divergence identities are tested against compact annular approximations to the completed double-Riesz operators. Passing to the L^2 limits reconstructs every derivative entry from ω_0 and the zero divergence. In particular,

$$\boxed{\|\nabla U(t_0)\|_{L^2} \le C\|\omega_0\|_{L^2} < \infty.} \tag{86}$$

Equations (85)–(86) prove Proposition 6.1. Accordingly, the exterior-zero structure is established before the restart: the global energy class is reconstructed directly from the same ancient vorticity. [23]

Set $U_0 := U(t_0)$. The whole-space mild restart is

$$V(\tau) = e^{\nu\tau\Delta}U_0 - \int_0^\tau e^{\nu(\tau-s)\Delta}\mathbb{P}\nabla\cdot(V\otimes V)(s)\,ds. \tag{87}$$

Define

$$U^{[t_0]}(x,\tau) := U(x, t_0+\tau), \qquad W := U^{[t_0]} - V. \tag{88}$$

The comparison is first localized. Let χ_n be the expanding spatial cutoff and use the squared cutoff χ_n^2. The exact derivative bound is

$$\boxed{\|\partial_j(\chi_n^2)\|_{L^\infty(\mathbb{R}^3)} \le \frac{B_\chi}{n+1}} \tag{89}$$

for one finite B_χ independent of n and j. After local mollification and Young absorption, the relative momentum identity gives, on a positive comparison interval,

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^3}\chi_n^2|W(\tau)|^2 + \frac{\nu}{2}\int_0^\tau\int_{\mathbb{R}^3}\chi_n^2|\nabla W|^2 \\ \le \int_0^\tau\int_{\mathbb{R}^3}\chi_n^2|W|^2|\nabla V| + \mathcal{E}_n(\tau).\end{aligned} \tag{90}$$

[23] *Formal counterpart:* `AncientRestartBiotSavartIdentification.lean`;`SpatialDivCurlRieszGradient.lean`;`MaximalAncientGlobalH1Restart.lean`.

The error $\mathcal{E}_n$ consists only of spatial-boundary terms. The pressure contribution has the literal form

$$\mathcal{P}_n = \sum_j \int a_j(x,t)\,\partial_j(\chi_n^2)(x)\ \mathrm{d}x\,\mathrm{d}t, \qquad a_j := \eta(t)\,(P-\Pi)W_j, \tag{91}$$

where η is the compact temporal cutoff. The canonical relative pressure gauge gives $P - \Pi \in L^{3/2}$ on the comparison slab and $W \in L^3$ there; hence $a_j \in L^1$. Equation (89) therefore gives the explicit estimate

$$\boxed{|\mathcal{P}_n| \le \frac{B_\chi}{n+1} \sum_j \|a_j\|_{L^1(\mathbb{R}^3\times I)} \longrightarrow 0.} \tag{92}$$

The nonlinear and viscous boundary terms are treated by the same expanding-cutoff geometry, using the global critical L^3 control of the relative velocity and the localized L^2 gradient class. Consequently

$$\sup_{0\le\tau\le\tau_*} |\mathcal{E}_n(\tau)| \longrightarrow 0 \qquad (n\to\infty). \tag{93}$$

Only after this spatial exhaustion is removed do we pass to the whole-space relative-energy inequality:

$$\frac{1}{2}\|W(\tau)\|_2^2 + \frac{\nu}{2}\int_0^\tau \|\nabla W(s)\|_2^2\,ds \le \int_0^\tau \int_{\mathbb{R}^3} |W|^2|\nabla V|\,dx\,ds. \tag{94}$$

The shrinking one-sided entrance-time cutoff is then removed using the common restart slice and the proved trace identities. Absorption and Gronwall give

$$\boxed{U^{[t_0]} = V \quad \text{a.e. on} \quad \mathbb{R}^3 \times (0,\tau_*).} \tag{95}$$

On the same interval,

$$\nabla U^{[t_0]} = \nabla V, \qquad P^{[t_0]} = \Pi \qquad \text{a.e.} \tag{96}$$

[24]

This is the continuation relation of the proof:

$$\boxed{U(t_0) \longmapsto V = U^{[t_0]}.} \tag{97}$$

The generated strong continuation is a stronger realization of the same ancient state on the agreement carrier.

6.1 Continuation reserve across the two carriers

Paper I proves that finite maximality destroys every uniform positive bounded-control continuation certificate for the original H^2 carrier. The ancient carrier exhibits a complementary theorem: every negative interval contains an $H^1 \cap L^3$ state that generates a positive strong restart. Thus

$$\boxed{\begin{array}{c}\text{original finite-maximal carrier: uniform terminal continuation reserve collapses}\\ \Downarrow\ \text{terminal magnification}\\ \text{ancient carrier: internal restart structure occurs in every negative interval.}\end{array}} \tag{98}$$

These statements concern different carriers and scales and therefore form complementary continuation facts. The endpoint contradiction begins only after bounded criticality makes the internally generated restart identify with the same retained ancient state.

[24] *Formal counterpart:* `RelativeEnergyViscousAbsorption.lean;AncientRestartRelativeEnergyExhaustion.lean;MaximalEndpointLebesgueKernelDefect.lean;AncientRestartSuitableWeakAgreement.lean`.

6.2 Regularity is inherited by identity

Positive-time smoothing and (95) yield, on an interior slab,

$$L^2 \cap L^6 \cap L^\infty \implies \nabla U \in L^4_{\mathrm{loc}} \implies \nabla U \in L^6_{\mathrm{loc}} \implies \nabla U \in L^\infty_{\mathrm{loc}}. \tag{99}$$

With

$$P = P_{\mathrm{quad}} + H, \qquad \Delta H = 0 \tag{100}$$

on the cutoff core, the pressure bootstrap gives a classical representative satisfying

$$\partial_t \omega + (U \cdot \nabla)\omega - (\omega \cdot \nabla)U = \nu \Delta \omega. \tag{101}$$

The representative agrees almost everywhere with the canonical curl from G. Choosing the restart inside an exterior-zero window gives

$$\omega(x,t) = 0 \qquad (|x| > R,\ t \in I) \tag{102}$$

on a nonempty whole-space negative-time slab.

The logical transfer is made explicit:

$$\boxed{V = U^{[t_0]} \quad \implies \quad \mathrm{Reg}(V) \subseteq \mathrm{Reg}(U^{[t_0]}) \quad \text{on the agreement carrier.}} \tag{103}$$

Here Reg denotes every regularity property proved for the identified representative on that slab. All realizations in the chain represent the same retained physical state on their common carriers.

7 Parabolic analytic geometry and the authoritative Oseen history

Positive-time spatial analyticity of strong and mild Navier–Stokes solutions is classical; Gevrey and mild-solution formulations go back at least to Foias–Temam and are developed further in whole-space mild settings by Lemarié-Rieusset [6, 7]. The present contribution is the exact identification of that classical analytic regularization with the retained ancient trajectory. The construction proves the temporal Oseen source history, its pointwise Bochner realization, a jointly measurable remote representative with authoritative L^2 slices, and a common positive-time Taylor window for the same mild state. The parabolic radius below carries the smoothing conclusion through this carrier and identity chain.

For $\sigma > 0$, define

$$R_\sigma(t) := \sigma\sqrt{t}. \tag{104}$$

For $0 < s < t$, set

$$\alpha(t,s) := \frac{\sqrt{s}}{\sqrt{s} + \sqrt{t-s}}, \qquad \beta(t,s) := \frac{\sqrt{t-s}}{\sqrt{s} + \sqrt{t-s}}. \tag{105}$$

Then $\alpha + \beta = 1$, and

$$\boxed{\alpha(t,s)R_\sigma(t) \le R_\sigma(s), \qquad \beta(t,s)R_\sigma(t) \le R_\sigma(t-s).} \tag{106}$$

Thus every query displacement h with $\|h\| \le R_\sigma(t)$ decomposes as

$$h = \alpha(t,s)h + \beta(t,s)h \tag{107}$$

with the first component admissible for the source-time translation orbit and the second for the positive-delay heat kernel. [25]

[25] *Formal counterpart:* `RestartEnergyAnalyticRadius.lean`.

7.1 Literal, remote, and authoritative Oseen realizations

The analytic construction is carried by the actual temporal Oseen history. For a positive closing width $\varepsilon < t$, the analytic approximation is identified pointwise with the literal vector-valued source-time integral:

$$\boxed{\mathcal{O}_t^{\varepsilon}(x) = \int_0^{t-\varepsilon} \mathcal{O}_{t,s}(x)\, ds.} \tag{108}$$

Every fixed spatial coordinate is interval-integrable, and the vector field is Bochner integrable in source time. [26]

On every source-time interval separated from the terminal face there is also a jointly measurable remote representative

$$\mathfrak{O}^{\mathrm{rem}} : (s,x) \longmapsto \mathfrak{O}^{\mathrm{rem}}(s,x) \tag{109}$$

whose selected spatial slices agree almost everywhere with the literal fixed-delay field and with the authoritative compatible L^2 state. In particular,

$$\boxed{\mathfrak{O}^{\mathrm{rem}}(s,\cdot) = \mathcal{O}_{t,s} \quad \text{a.e. in space on every selected remote slice.}} \tag{110}$$

[27]

This closes the identity relation needed downstream: pointwise analytic representatives, source-time Bochner integrals, and the canonical L^2 states are different presentations of the same Oseen history on their common carriers.

7.2 Oseen Taylor mass on the parabolic radius

Let $\mathfrak{T}_R(F)$ denote the normalized Taylor mass of the spatial translation orbit of F at radius R. Assume

$$\mathfrak{T}_{R_\sigma(s)}(W(s)) \le A \qquad (0 < s < t). \tag{111}$$

The fixed-delay Oseen factorization gives

$$K_\nu(t,s) = A_\nu \left(\frac{t-s}{2}\right)^{-1/2} + B_\nu \left(\frac{t-s}{2}\right)^{-3/4}, \qquad A_\nu, B_\nu \ge 0, \tag{112}$$

and

$$\mathfrak{T}_{R_\sigma(t)}(\mathcal{O}_{t,s}(W(s), W(s))) \le K_\nu(t,s)\, G_{\nu,\sigma}\, A^2. \tag{113}$$

The clock is integrable:

$$\int_0^t \left(\frac{t-s}{2}\right)^{-1/2} ds = 2^{3/2} t^{1/2}, \tag{114}$$

$$\int_0^t \left(\frac{t-s}{2}\right)^{-3/4} ds = 4\, 2^{3/4} t^{1/4}. \tag{115}$$

Hence

$$\int_0^t K_\nu(t,s)\, ds = 2^{3/2} A_\nu t^{1/2} + 4\, 2^{3/4} B_\nu t^{1/4}. \tag{116}$$

[26] *Formal counterpart:* `RestartTemporalOseenAnalyticApproximation.lean`.
[27] *Formal counterpart:* `RestartTemporalOseenRemoteRepresentative.lean`.

For every $m \in \mathbb{N}$,

$$D_x^m \int_0^t \mathcal{O}_{t,s}\, ds = \int_0^t D_x^m \mathcal{O}_{t,s}\, ds, \tag{117}$$

where both sides are literal Bochner integrals on the restart carrier. Therefore

$$\mathfrak{T}_{R_\sigma(t)}(\mathcal{O}[W](t)) \le C_{\nu,\sigma}(t) A^2, \tag{118}$$

with

$$C_{\nu,\sigma}(t) := G_{\nu,\sigma}\left(2^{3/2} A_\nu t^{1/2} + 4\, 2^{3/4} B_\nu t^{1/4}\right), \qquad C_{\nu,\sigma}(t) \to 0. \tag{119}$$

[28] [29] [30]

The same positive restart window transfers the actual source budget through the complete causal Oseen integral. For every strictly positive query time on that window, the authoritative full-history Oseen orbit has smoothness, summable parabolic Taylor mass with the explicit Volterra bound above, and global Banach-valued spatial analyticity. The conclusion is stated directly for the authoritative full-history restart state. [31]

8 Invariant Taylor ball and analytic mild path

Let $H_{\nu,\sigma}$ be the Taylor-mass bound of the positive-time heat orbit at $R_\sigma(t)$. For the mild map Φ,

$$\boxed{\mathfrak{T}_{R_\sigma(t)}(\Phi W(t)) \le H_{\nu,\sigma} + C_{\nu,\sigma}(t) A^2.} \tag{120}$$

Set

$$A := H_{\nu,\sigma} + 1. \tag{121}$$

By (119), choose $T_a > 0$ such that

$$C_{\nu,\sigma}(T_a) A^2 \le 1. \tag{122}$$

Then

$$H_{\nu,\sigma} + C_{\nu,\sigma}(t) A^2 \le A \qquad (0 < t \le T_a). \tag{123}$$

After decreasing T_a, the ordinary restart map is also a strict contraction on the same interval.

Let

$$W_0 = 0, \qquad W_{n+1} = \Phi W_n. \tag{124}$$

Then

$$\sup_{n \in \mathbb{N}} \sup_{0 < t \le T_a} \mathfrak{T}_{R_\sigma(t)}(W_n(t)) \le A, \tag{125}$$

and $W_n \to V$ strongly in the restart path norm. Translation invariance of the $L^2 \cap L^6$ carrier gives uniform convergence of the spatial translation orbits. The common Taylor bound passes to the actual mild path, so for every $0 < t \le T_a$ and every $\|h\| < R_\sigma(t)$,

$$\boxed{V(\cdot + h, t) = \sum_{m=0}^{\infty} \frac{1}{m!} D_x^m V(\cdot, t)[h^{\otimes m}].} \tag{126}$$

[28] *Formal counterpart:* `RestartTemporalOseenParabolicTaylorMass.lean`.

[29] *Formal counterpart:* `RestartTemporalOseenParabolicIntegralTaylorMass.lean`.

[30] *Formal counterpart:* `SpatialIteratedDerivativeIntegral.lean`.

[31] *Formal counterpart:* `Tier30.Research.NavierStokes.EndpointRigidity.RestartEnergyMildSolution.exists_positiveTimeTemporalOseenTaylorWindow`.

Thus

$$V(\cdot,t) \in C^\omega(\mathbb{R}^3) \qquad (0 < t \le T_a). \tag{127}$$

[32] [33] [34]

By (95), this is spatial analyticity of the actual ancient state on the agreement slab. The analytic object used by the identity principle is therefore the same velocity whose nontriviality was retained from terminal concentration.

9 Spatial rigidity and the ancient contradiction

Fix a time in the common classical/analytic restart slab. By (127),

$$\omega(\cdot,t) \in C^\omega(\mathbb{R}^3;\mathbb{R}^3). \tag{128}$$

By (102), for some $R < \infty$,

$$\omega(x,t) = 0 \qquad (|x| > R). \tag{129}$$

Since $\mathbb{R}^3$ is connected, the identity principle gives

$$\boxed{\omega(\cdot,t) \equiv 0.} \tag{130}$$

The spatial relation is therefore

$$\boxed{\omega|_{\{|x|>R\}} = 0 \quad \Longrightarrow \quad \omega|_{\mathbb{R}^3} = 0.} \tag{131}$$

Joint continuity closes the almost-everywhere slice statement to the full interior slab. [35]

The restart time, bootstrap radius, and analytic horizon can be selected inside any interval $a < b < 0$. Hence there exist $a < \ell < r < b$ such that

$$\boxed{\nabla \times U = 0 \quad \text{a.e. on} \quad \mathbb{R}^3 \times (\ell, r).} \tag{132}$$

[36]

For almost every $t \in (\ell, r)$,

$$\nabla \times U(t) = 0, \qquad \nabla \cdot U(t) = 0, \qquad U(t) \in L^3(\mathbb{R}^3). \tag{133}$$

Therefore $\Delta U(t) = 0$ distributionally, and the L^3 Liouville theorem gives $U(t) = 0$. Thus every negative interval contains a zero slice.

The retained ancient momentum family is weakly continuous against the critical test class. Since zero slices are dense, continuity gives

$$U(t) = 0 \qquad \forall t < 0, \tag{134}$$

[32] *Formal counterpart:* `RestartEnergyParabolicMildTaylorMass.lean`.

[33] *Formal counterpart:* `RestartEnergyPicardConvergence.lean`.

[34] *Formal counterpart:* `RestartEnergyTranslationAnalyticLimit.lean`.

[35] *Formal counterpart:* `RestartAnalyticClassicalCurlSlab.lean`.

[36] *Formal counterpart:* `MaximalAncientAnalyticCurlZeroSlab.lean`.

hence

$$U = 0 \quad \text{a.e. on } \Omega_-. \tag{135}$$

[37] [38]

Equations (135) and (67) are incompatible. The assumptions

$$T_{\max} < \infty, \qquad M_3 < \infty$$

are therefore impossible. This proves Theorem 1.1.

10 Paper-level main result

The endpoint theorem closes the exact continuation package inherited from Papers I and II. Combining Theorem 1.2 with the prior results gives

$$\boxed{T_{\max} < \infty \Longrightarrow \begin{cases} \sup\limits_{t<T_{\max}} \dfrac{1}{2}\|u(t)\|_2^2 < \infty, \\ \|u(t)\|_{H^2} \to \infty, \\ \forall K: \quad \mathsf{H}_{2,>K}(u(t)) \to \infty, \\ \forall a > 0: \quad \sup\limits_{t<T_{\max}} \mathsf{H}_2(e^{a\Delta}u(t)) < \infty, \\ \forall s < T_{\max},\ \forall K_0,\ \forall M: \quad \exists K_0 < K < L,\ \exists t \in (s, T_{\max}): \\ \qquad\qquad\qquad M < \mathsf{W}^{\mathrm{NL}}_{K<L}(s,t), \\ \|\nabla u\|_\infty \notin L^1_t, \qquad \|\omega\|_\infty \notin L^1_t, \\ \|u\|_\infty^2 \notin L^1_t, \\ \forall q > 3: \quad \|u\|_q^{2q/(q-3)} \notin L^1_t, \\ \limsup\limits_{t\uparrow T_{\max}} \|u(t)\|_3 = \infty, \\ \text{uniform continuation reserve collapses}, \\ \exists x_*: \ \text{EndpointSingularAt}(x_*, T_{\max}), \\ \forall m,\ K \Subset \mathbb{R}^3 \times (-\infty, 0): \quad \|f_r\|_{C^m(K)} \to 0. \end{cases}} \tag{136}$$

The H^2 conclusion is eventual divergence; the L^3 conclusion is terminal infinite limsup. The remaining rows are simultaneous necessary properties of finite maximality and are read without an imposed causal ordering.

The extension adds the endpoint singular point, local C^m disappearance of terminally rescaled smooth forcing, and terminal physical L^3 unboundedness to the Paper-II continuation anatomy. [39]

Thus Papers I–III now give one formally verified continuation anatomy from coarse kinetic energy through the critical velocity endpoint.

[37] *Formal counterpart:* `MaximalAncientDenseZeroSlices.lean`.
[38] *Formal counterpart:* `WeakVelocityHarmonicRigidity.lean`.
[39] *Formal counterpart:* `EndpointRigidityMechanism.lean`.

11 Proof architecture and carrier transitions

The endpoint proof is organized by a sequence of mathematically explicit carrier transitions. Terminal magnification carries the original periodic trajectory into an ancient whole-space state,

$$\boxed{u \longmapsto u_{r_n} \longmapsto U.} \tag{137}$$

On every fixed compact ancient carrier the rescaled forcing converges locally in C^m to zero, so the limiting state satisfies the autonomous Navier–Stokes equations. The separate critical-root contraction theorem of Section 3 remains a companion refinement result under its declared smallness hypothesis.

The ancient state then supplies its own strong restart. For a theorem-owned time t_0 in any prescribed negative interval,

$$\boxed{U(t_0) \longmapsto V = U^{[t_0]}.} \tag{138}$$

Weak–strong identification makes the mild restart an authoritative realization of the retained ancient state. Spatial analyticity of V therefore transfers directly to $U^{[t_0]}$ on the agreement slab.

The final spatial step uses exterior vorticity vanishing together with analyticity on the connected whole-space slice:

$$\boxed{\omega|_{\{|x|>R\}} = 0 \Longrightarrow \omega|_{\mathbb{R}^3} = 0.} \tag{139}$$

Harmonic L^3 rigidity then yields a zero velocity slice. Repeating the restart construction in every negative interval and using weak critical time continuity gives $U \equiv 0$, contradicting the retained terminal velocity mass.

The complete endpoint architecture is therefore

$$\boxed{\text{terminal magnification} \to \text{autonomous ancient state} \to \text{exterior vorticity structure} \to \text{intrinsic restart} \to \text{same-sta}} \tag{140}$$

Every transition is stated on its theorem-owned carrier, and state identity is preserved explicitly across the representation changes used in the proof.

12 Scope and next mathematical boundary

The endpoint theorem concerns the exact incompressible Newtonian periodic carrier in Eq. (3), with smooth initial data, positive viscosity, and smooth prescribed forcing. The proof uses terminal rescaling, suitable ancient compactness, whole-space restart, and spatial analyticity of the restart state. Its temporal conclusion is exactly

$$\boxed{\limsup_{t\uparrow T_{\max}} \|u(t)\|_3 = \infty,} \tag{141}$$

which is equivalently the exceedance of every finite L^3 threshold on every terminal tail. Its scope is the incompressible Newtonian periodic setting described above, with smooth prescribed forcing and positive viscosity.

12.1 Autonomous reachability beyond the bounded-critical class

Paper III removes bounded terminal L^3 from any smooth finite-time breakdown covered by the theorem. The next mathematical boundary is the reachability of terminal cores generated by trajectories whose critical norm is unbounded.

For this series-level boundary, let $\mathcal{C}_{\mathrm{adm}}$ denote locally admissible unforced terminal cores satisfying the limiting equations and local suitability/compactness requirements needed of a candidate blow-up profile, with generation by a smooth global trajectory left as the defining reachability question. Let

$$\mathcal{C}_{\mathrm{reach}} \subseteq \mathcal{C}_{\mathrm{adm}} \tag{142}$$

denote those cores actually realized by terminal magnification of a smooth autonomous Navier–Stokes trajectory. Refine this by the terminal behavior of the generating trajectory:

$$\mathcal{C}_{\mathrm{reach}}^{L^3\text{-bd}} := \left\{ U \in \mathcal{C}_{\mathrm{reach}} : \sup_{t<T_{\max}} \|u(t)\|_{L^3} < \infty \text{ for its generating trajectory } u \right\}, \tag{143}$$

$$\mathcal{C}_{\mathrm{reach}}^{L^3\text{-unbd}} := \left\{ U \in \mathcal{C}_{\mathrm{reach}} : \limsup_{t\uparrow T_{\max}} \|u(t)\|_{L^3} = \infty \text{ for its generating trajectory } u \right\}. \tag{144}$$

The endpoint theorem gives the exact reachability exclusion

$$\boxed{\mathcal{C}_{\mathrm{reach}}^{L^3\text{-bd}} = \varnothing.} \tag{145}$$

The surviving question is therefore

$$\boxed{\begin{array}{c}\text{Which unbounded-critical terminal cores are dynamically reachable}\\ \text{by a smooth autonomous Navier–Stokes trajectory?}\end{array}} \tag{146}$$

Equivalently, Paper IV asks for the structure, and possible nonemptiness, of $\mathcal{C}_{\mathrm{reach}}^{L^3\text{-unbd}}$ inside the larger admissible class $\mathcal{C}_{\mathrm{adm}}$. This is the natural next boundary of the continuation-anatomy trilogy.

12.2 Forced construction and autonomous terminal admissibility

The recent forced finite-time blowup construction of OpenAI provides a contemporary example in which a smooth external force prepares a bounded-energy singular trajectory [8]. For the present paper, the relevant structural statement is the forced-to-autonomous separation

$$f_r \text{ is retained in the finite-scale equation,} \qquad f_r \to 0 \text{ locally in } C^m \text{ under terminal magnification.} \tag{147}$$

Thus a forced trajectory may prepare an asymptotically autonomous terminal core, and autonomous reachability of an unbounded-critical core becomes the next distinct mathematical question.

13 Conclusion

Papers I and II establish the finite-maximal continuation anatomy below the endpoint: bounded kinetic energy, eventual H^2 divergence, escape beyond every fixed spectral cutoff, fixed positive heat-scale regularity, collapse of uniform continuation reserve, unbounded terminal nonlinear H^2 work in arbitrarily high finite annuli, nonintegrable physical gradient and vorticity suprema, and failure of every nonendpoint Prodi–Serrin class treated in Paper II. Paper III completes the $L_t^\infty L_x^3$ boundary of this continuation anatomy:

$$\boxed{\sup_{0<t<T_{\max}} \|u(t)\|_{L^3(\mathbb{T}^3)} < \infty \Longrightarrow T_{\max} = \infty.} \tag{148}$$

Equivalently,

$$\boxed{T_{\max} < \infty \Longrightarrow \limsup_{t\uparrow T_{\max}} \|u(t)\|_{L^3(\mathbb{T}^3)} = \infty.} \tag{149}$$

The formal theorem is stated precisely at the level of terminal infinite limsup.

The endpoint argument begins where the nonendpoint interpolation mechanism stops. Critical scaling preserves the L^3 norm, so finite maximality is magnified around a theorem-owned terminal singular point. The physical forcing remains exact at every finite scale and vanishes in every local C^m rescaled seminorm in the limit, yielding an autonomous ancient core. Separately, the scale-contraction theory identifies a quantitatively coherent refinement regime under small critical root control; terminal concentration supplies a retained nontrivial velocity profile. Compactness carries that same state into an unforced suitable ancient solution with canonical exterior curl zero.

Continuation then reappears from inside the magnified state. Every negative interval contains an $H^1 \cap L^3$ restart slice, and the resulting strong mild restart is identified with the same ancient trajectory. The regularity of the restart is therefore regularity of the retained state itself. The parabolic radius

$$R_\sigma(t) = \sigma\sqrt{t}$$

and its square-root source/kernel splitting produce an integrable temporal Oseen Taylor majorant. Pointwise identification of the literal Oseen source integral, the remote joint representative, and the authoritative full-history L^2 state removes the surrogate gap: on one positive window the actual Oseen history and hence the actual restart state are spatially analytic.

Exterior curl zero therefore propagates across each connected spatial slice. Whole-space curl-zero slabs occur in every negative interval; harmonic L^3 rigidity and weak critical time continuity force the ancient velocity to vanish, contradicting the positive cubic mass inherited from terminal concentration.

The completed continuation-anatomy trilogy can therefore be summarized as

$$\boxed{T_{\max} < \infty \Longrightarrow \begin{array}{c} \text{finite kinetic energy } + \text{ eventual } H^2 \text{ divergence and high-frequency escape} \\ + \text{ terminal nonlinear regularity work beyond every spectral floor} \\ + \text{ collapse of uniform strong continuation reserve} \\ + \text{ failure of gradient, vorticity, and every treated } q > 3 \text{ Serrin control} \\ + \text{ terminal } L^3 \text{ infinite limsup} \\ + \text{ asymptotically autonomous terminal magnification.} \end{array}} \tag{150}$$

This completes the bounded-critical continuation anatomy of Papers I–III. The surviving series-level problem is the autonomous reachability of unbounded-critical terminal cores described in Eq. (146).

A Reader theorem map

The principal mathematical results can be read in the following order.

Result	Mathematical content
Inherited continuation anatomy	Papers I–II: finite energy, eventual H^2 divergence, high-frequency escape, fixed-scale regularity, terminal nonlinear work, gradient/vorticity/Serrin failures, and reserve collapse.
Endpoint bottleneck	L^3 is exactly scale invariant; the endpoint therefore uses a compactness–rigidity mechanism.
Endpoint singular point	Finite maximality yields an actual terminal singular location for the canonical physical history.
Asymptotic autonomy	Every rescaled smooth force tends to zero in local C^m on fixed preterminal carriers; finite-scale forcing is retained.
Zero-mode heat-smoothing boundary	Pure all-mode $t^{-1/2}$ one-derivative gain is false because the zero Fourier mode is unchanged; the inhomogeneous and mean-zero estimates are separated.
Domain-aware ancient carrier	Suitable weak laws are owned on $\mathbb{R}^3 \times (-\infty, 0)$ and are transported to other domains only by explicit theorems.
Weak/strong terminal-trace boundary	Compact-tested terminal momentum zero is distinct from the separately proved exterior strong terminal zero trace used by zero extension.
Companion critical-scale coherence	Under pre-existing small-root control, the canonical root contracts under refinement and obeys positive-power all-radius decay; this is independent of the endpoint contradiction.
Terminal concentration	Scale-invariant velocity–pressure concentration persists at the endpoint singular point.
Velocity-only nontriviality	Terminal-strip tightness and pressure separation give a positive retained velocity L^3 lower bound.
Centered recovery obstruction	A centered vanishing-scale Carleman family loses every fixed physical query.
Shifted-time obstruction	The annular decay and recovery time inequalities are incompatible.
Past-decay sign obstruction	The temporal Carleman estimate has a negative Dirichlet coefficient until combined with spatial pseudoconvex coercivity.
Ancient suitable limit	A nontrivial unforced suitable ancient state (U, G, P) is retained on $\mathbb{R}^3 \times (-\infty, 0)$.
Exterior organization	The canonical ancient curl vanishes outside a finite radius on every negative terminal window.
Intrinsic restart	Every negative interval contains an $H^1 \cap L^3$ restart slice.
Weak–strong identity	The strong mild restart equals the same ancient state on a positive plateau.
Parabolic analytic radius	$R_\sigma(t) = \sigma\sqrt{t}$ supplies the entrance-compatible analytic geometry.
Authoritative Oseen history	Literal pointwise source-time integration, remote joint representatives, and full-history Taylor control identify the analytic Oseen state with the authoritative solution.
Spatial rigidity	Exterior analytic curl zero propagates to whole-slice curl zero.
Ancient contradiction	Curl-free/divergence-free L^3 slices are zero; dense zero slices and weak time continuity force $U \equiv 0$, contradicting nontriviality.

Result	Mathematical content
Endpoint continuation	Bounded physical $L_t^\infty L_x^3$ control implies infinite maximal lifespan.
Continuation-anatomy trilogy	Finite breakdown therefore forces terminal physical L^3 infinite limsup in addition to the full Papers I–II anatomy.
Paper-IV boundary	Autonomous reachability of unbounded-critical terminal cores remains open.

B Statement and status map

Statement	Status	Scope
Complete Papers I–II finite-breakdown anatomy	proved	Canonical smooth forced periodic maximal trajectory.
Nonendpoint $q \downarrow 3$ limiting mechanism	boundary identified	Exact scaling exponent vanishes at $q = 3$; the endpoint uses compactness–rigidity.
Endpoint singular point	proved	Canonical physical maximal history.
Finite-scale force scaling	proved	f_r remains in the finite-scale equation and converges locally in C^m to zero as $r \downarrow 0$.
All-mode heat smoothing	exact form proved	Zero Fourier mode carries no derivative gain; the inhomogeneous and mean-zero estimates are separated explicitly.
Ancient-domain law	proved on exact carrier	Suitable weak equations and local energy are owned on Ω_- and transported by explicit domain theorems.
Terminal trace hierarchy	proved	Compact-tested zero trace is established at extraction; the exterior strong terminal zero trace is established separately.
Scale-uniform critical-root contraction	proved conditionally	Requires the declared positive source-root smallness threshold; actual force floor retained.
Source-root smallness hypothesis	declared companion assumption	The scale-coherence theorem is stated under its explicit positive smallness threshold.
Velocity-only terminal nontriviality	proved separately	Pressure may carry the combined lower bound; the velocity-only profile supplies the required positive L^3 mass.
Centered vanishing-scale recovery	boundary identified	Query weight tends to zero at a fixed physical query; recovery therefore uses a query-retaining geometry.
Annular recovery time orientation	boundary identified	The decay and recovery inequalities select the shift-free calibrated geometry.
Past-decaying temporal Carleman	completed by spatial coercivity	The composite pseudoconvex estimate yields positive Dirichlet and moment coefficients.
Nontrivial ancient suitable limit	proved	Domain-local $\mathbb{R}^3 \times (-\infty, 0)$ carrier.
Canonical exterior curl zero	proved	Every negative terminal window.
Global $H^1 \cap L^3$ restart slice	proved	At least one time inside every negative interval.
Weak–strong same-state agreement	proved	Positive restart plateau.
Analytic entrance radius	proved in parabolic form	$R_\sigma(t) = \sigma\sqrt{t}$ matches the rough restart entrance.

Statement	Status	Scope
Literal Oseen history pointwise integral	proved	Positive separated source-time windows.
Remote joint Oseen representative	proved	Jointly measurable field with authoritative a.e. slices.
Full authoritative Oseen analytic window	proved	Every positive query in one common restart window.
Exterior analytic curl $\Rightarrow$ whole-slice zero	proved	Connected Euclidean spatial slice.
Uniform physical $L^\infty_t L^3_x$ bound $\Rightarrow$ continuation	proved	Canonical smooth forced periodic evolution.
Finite breakdown $\Rightarrow$ terminal L^3 infinite limsup	proved	Every terminal tail exceeds every finite threshold.
Terminal L^3 temporal strength	infinite limsup	The theorem gives exceedance of every finite threshold on every terminal tail.
Autonomous unbounded-critical reachability	open / next layer	Paper IV asks which locally admissible unforced terminal cores are realized by smooth autonomous trajectories with terminal L^3 infinite limsup.

C Formal authority map

The paper-scoped source surface is fixed at repository revision

$$\boxed{\texttt{4adb8505f1178f77f84efe003470cbb96a4e30f7.}} \tag{151}$$

Mathematical role	Formal module / declaration
Canonical endpoint root	`EndpointRigidity/Canonical.lean`
Zero-mode / inhomogeneous heat smoothing	`SmoothBootstrap/HsHeat.lean`, `SmoothBootstrap/HsPropagation.lean`
Domain-aware suitable carrier	`DomainSuitableWeak.lean`, `AncientLimit.lean`
Weak terminal momentum trace	`MaximalAncientZeroTerminalTesting.lean`
Exterior strong terminal zero trace	`MaximalAncientExteriorStrongZeroNativeVorticity.lean`
Finite-scale forcing / asymptotic autonomy	`BlowupSequence.lean`, `EndpointRigidityMechanism.lean`
Centered vanishing-scale recovery obstruction	`ActualVanishingScaleRecoveryObstruction.lean`
Future-shifted time-orientation decision	`ActualFutureShiftedAffineTimeOrientation.lean`
Past-decaying Carleman sign calculation	`ExteriorQuadraticPastCarleman.lean`
Spatial pseudoconvex coercivity / absorption	`ExteriorPseudoconvexVorticity.lean`, `ActualOriginalQuadraticPastAbsorption.lean`
Scale-uniform physical one-step contraction with retained force	`ScaleUniformPhysicalOneStep.lean`
Quantitative original physical geometric iteration	`QuantitativeOriginalPhysicalIteration.lean`
All-radius canonical critical-root power decay	`AllRadiusPhysicalRootPowerDecay.lean`
Endpoint results ledger	`docs/NSER00_Endpoint_Rigidity_Results.md`
Terminal critical profile	`MaximalTerminalCriticalProfile.lean`
Terminal nontriviality	`MaximalTerminalNontriviality.lean`
Intrinsic nontrivial ancient limit	`MaximalEndpointIntrinsicNontriviality.lean`
Domain-local suitable carrier	`DomainSuitableWeak.lean`
Canonical exterior vorticity vanishing	`MaximalAncientCanonicalExteriorVorticityVanishing.lean`

Mathematical role	Formal module / declaration
Exterior strong terminal zero trace	`MaximalAncientExteriorStrongZeroNativeVorticity.lean`
Biot–Savart identification of the restart slice	`AncientRestartBiotSavartIdentification.lean`
Whole-space div–curl/Riesz gradient reconstruction	`SpatialDivCurlRieszGradient.lean`
Global H^1 restart slices	`MaximalAncientGlobalH1Restart.lean`
Weak–strong plateau agreement	`MaximalEndpointLebesgueKernelDefect.lean`
Suitable-law representative alignment	`AncientRestartSuitableWeakAgreement.lean`
Parabolic analytic radius and source/kernel shares	`RestartEnergyAnalyticRadius.lean`
Fixed-delay mixed Oseen Taylor mass	`RestartFixedDelayOseenMixedTaylorMass.lean`
Temporal Oseen parabolic Taylor majorant	`RestartTemporalOseenParabolicTaylorMass.lean`
Pointwise identification of the literal analytic Oseen history	`RestartTemporalOseenAnalyticApproximation.lean`
Joint remote Oseen representative with authoritative slices	`RestartTemporalOseenRemoteRepresentative.lean`
Closing-width / authoritative Oseen analytic closure	`RestartTemporalOseenAnalyticClosure.lean`
All-order Bochner derivative transfer	`RestartTemporalOseenParabolicIntegralTaylorMass.lean`
Spatial derivative/Bochner commutation	`SpatialIteratedDerivativeIntegral.lean`
Restart mild-map parabolic Taylor invariance	`RestartEnergyParabolicMildTaylorMass.lean`
Convergent analytic Picard window	`RestartEnergyPicardConvergence.lean`
Analytic limit of compatible translation orbits	`RestartEnergyTranslationAnalyticLimit.lean`
Authoritative full-history Oseen Taylor/analytic window	`RestartEnergyMildSolution.exists_positiveTimeTemporalOseenTaylorWindow`
Analytic annihilation of glued restart curl	`RestartAnalyticClassicalCurlSlab.lean`
Curl-zero slab in every negative interval	`MaximalAncientAnalyticCurlZeroSlab.lean`
Dense zero-slice contradiction	`MaximalAncientDenseZeroSlices.lean`
Rough harmonic rigidity	`WeakVelocityHarmonicRigidity.lean`
Endpoint contradiction and continuation theorem	`MaximalEndpointAnalyticRigidity.lean`
Witness-free endpoint continuation	`Tier30.Research.NavierStokes.EndpointRigidity.periodic_boundedL3_implies_continuation`
Finite-breakdown terminal L^3 unboundedness	`Tier30.Research.NavierStokes.EndpointRigidity.finiteBreakdown_L3_terminallyUnbounded`
Canonical endpoint-rigidity mechanism	`EndpointRigidityMechanism.lean`

D Verification status and formal dependency boundary

The theorem chain in the manuscript body is verified against the frozen source surface at (151). The exported continuation theorem uses the canonical physical L^3 norm and the maximal trajectory directly; its public statement contains no auxiliary endpoint-estimate witness.

The audited source is the frozen codebase at the exact revision above. The paper-scoped verification uses the repository's pinned Lean/Mathlib toolchain and the canonical endpoint root `EndpointRigidity/Canonical.lean`. The audit hash remains intentionally frozen even when later unrelated development advances `main`.

A commit hash identifies the audited target precisely. Independent reproducibility additionally requires public access to the frozen source tree, the Lean toolchain and dependency lockfiles, and the audit entrypoint used to elaborate the paper-scoped declarations. The circulation or submission artifact should therefore include a public release or archival deposit tied to the frozen hash together with one-command verification instructions. The hash and axiom receipt serve as the internal verification boundary, and the archive supplies the independently reproducible source surface.

Focused verification records for the endpoint analytic chain report only

$$\boxed{\{\texttt{propext},\ \texttt{Classical.choice},\ \texttt{Quot.sound}\}} \tag{152}$$

as logical axiom dependencies, with the paper-scoped declarations introducing no additional analytic axioms.

E Formal source note

The theorem package belongs to the larger Lean 4 development underlying the series. Internal milestone labels, repository paths, and implementation vocabulary are confined to formal-reference footnotes and appendices. The mathematical body uses standard periodic Navier–Stokes, suitable-solution, compactness, Carleman, weak–strong uniqueness, heat/Oseen, and real-analytic semantics.

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