Pair-Generated Completion of Gaussian Physical States

Closed quadratic generators, faithful two-particle seeds, and nontruncatable current physicality in a machine-checked gauge-field model

Zed James

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Abstract

We study a three-parameter family of normalized Gaussian states selected by a closed matter-current constraint in a bosonic Fock representation over a finite-dimensional one-particle space. Their occupation coefficients obey an all-orders recurrence controlled by a symmetric pair kernel K_a, with $a = (a_1, a_2, a_3) \in (0, \infty)^3$. The same recurrence determines a positive one-particle precision G_a, related to the pair kernel by the Cayley transform

$$K_a = (I - G_a)(I + G_a)^{-1}.$$

Starting from this data, we construct the associated quadratic pair creator on the finite-occupation core and identify its closed realization with both the maximal square-summable coefficient action and the Hilbert adjoint of the reverse quadratic core operator.

The resulting operator reconstructs the Gaussian state one sector at a time,

$$P_{2n}\Psi_a = \frac{V_a}{n!}(\overline{A}_a^\dagger)^n\Omega, \qquad P_{2n+1}\Psi_a = 0,$$

where $V_a = \langle \Omega, \Psi_a \rangle > 0$. The first nonvacuum sector is already informative. After removing the vacuum coefficient,

$$\Xi_a := V_a^{-1} P_2 \Psi_a = \overline{A}_a^\dagger \Omega,$$

the map $a \mapsto \Xi_a$ is injective, and canonical two-particle coefficient functionals recover the entries of K_a up to fixed occupation normalizations. Thus the degree-two seed determines the finite Gaussian modulus even though it is only one sector of the completed state.

The orthogonal sum of the reconstructed sectors converges to Ψ_a. A coefficientwise exponential $\mathcal{E}_a$ gives a closed densely defined operator realization with

$$\Psi_a = V_a \, \mathcal{E}_a \Omega,$$

and spectral diagonalization yields

$$\|E_a(t)\|^2 = \det\bigl(I - |t|^2 K_a^* K_a\bigr)^{-1/2}, \qquad |t| \, \|K_a\| < 1,$$

together with the normalization identity

$$V_a^4 = \det(I - K_a^* K_a).$$

Finally, the finite pair truncations converge in norm to Ψ_a, while every nonzero truncation remains outside the exact joint current kernel. The Gaussian family is therefore governed and identified by finite pair data, whereas exact current physicality is realized only by the completed all-orders state.

1 Introduction

The companion matter-current analysis isolates a positive-orthant family of normalized Gaussian states whose exact current constraint excludes every nonzero vector of bounded particle number [1]. For

$$a = (a_1, a_2, a_3) \in (0, \infty)^3,$$

the Fock coefficients of Ψ_a obey an all-orders recurrence

$$c_s(n)\Psi_{a,n+e_s} = \sum_r K_a(s,r) b_r(n) \Psi_{a,n-e_r}. \tag{1}$$

The striking feature of Eq. (1) is that the same symmetric kernel appears at every occupation number. This suggests that the completed state may be controlled by a single quadratic law, but making that statement precise requires more than writing down the corresponding formal creation operator.

On the finite-occupation core we introduce

$$A^\dagger_{a,\mathrm{core}} = \frac{1}{2}\sum_{r,s} K_a(r,s) a_r^\dagger a_s^\dagger. \tag{2}$$

The first result identifies the operator selected by this expression. Its graph closure agrees with the maximal coefficient realization and with the Hilbert adjoint of the reverse quadratic core operator,

$$\overline{A^\dagger_{a,\mathrm{core}}} = A^\dagger_{a,\max} = (A_{a,\mathrm{core}})^*. \tag{3}$$

Here $A^\dagger_{a,\max}$ acts by the same coefficient formula on the largest domain for which the output remains square summable. The finite-occupation space is a graph core, so Eq. (3) fixes both the action and the domain of the closed pair creator.

This closed operator reproduces the state sector by sector:

$$P_{2n}\Psi_a = \frac{V_a}{n!}(\overline{A}_a^\dagger)^n \Omega, \qquad P_{2n+1}\Psi_a = 0, \tag{4}$$

with $V_a = \langle \Omega, \Psi_a \rangle > 0$. Since the sectors are orthogonal, their sum converges in Fock norm to the original Gaussian. The recurrence therefore admits an operator realization in which one closed quadratic generator produces the entire even-particle hierarchy.

The degree-two sector contains a second piece of information which becomes useful later in the series. Define

$$\Xi_a := V_a^{-1} P_2 \Psi_a = \overline{A}_a^\dagger \Omega. \tag{5}$$

The vector Ξ_a is nonzero and determines the Gaussian modulus. Indeed,

$$\Xi_a = \Xi_b \implies K_a = K_b \implies a = b. \tag{6}$$

The first implication follows from canonical coefficient readouts of the two-particle sector, while the second follows from the injectivity of the pair-kernel parametrization. Thus the same state exhibits two features that are often associated with rather different descriptions: its finite degree-two data are sufficient to identify the constitutive parameter, but the exact current constraint is satisfied only after the full even hierarchy is completed.

The algebraic geometry behind this construction is encoded by the positive precision G_a. The pair kernel is its Cayley transform,

$$K_a = (I - G_a)(I + G_a)^{-1}, \qquad G_a = (I - K_a)(I + K_a)^{-1}, \tag{7}$$

and K_a is self-adjoint and strictly contractive. These properties control both the operator theory and the normalization. Writing

$$E_a(t) := \sum_{n \geq 0} \frac{t^n}{n!} (\overline{A}_a^\dagger)^n \Omega,$$

one obtains

$$\|E_a(t)\|^2 = \det\Big(I - |t|^2 K_a^* K_a\Big)^{-1/2}, \qquad |t| \, \|K_a\| < 1, \tag{8}$$

and at the physical point

$$V_a^4 = \det(I - K_a^* K_a). \tag{9}$$

A coefficientwise exponential provides a closed operator realization of the same sum,

$$\Psi_a = V_a \, \mathcal{E}_a \Omega, \tag{10}$$

without identifying $\mathcal{E}_a$ with a functional-calculus exponential of the unbounded creator.

The role of completion is most transparent for the partial sums

$$S_N(a) := V_a \sum_{n=0}^{N} \frac{1}{n!} (\overline{A}_a^\dagger)^n \Omega. \tag{11}$$

They satisfy

$$S_N(a) \notin \mathcal{K}_{\text{joint}} \quad \forall N, \qquad S_N(a) \longrightarrow \Psi_a \in \mathcal{K}_{\text{joint}}. \tag{12}$$

Hence the completed vector is not being used as an approximation to a finite current-physical state; the infinite realization is part of the exact solution itself. All statements in this paper are made at the established finite regulator. We also record an exact isometric transport of the construction to a closed image in a completed momentum carrier, while regulator removal is left separate.

1.1 Relation to nearby work

Quadratic bosonic generators, squeezed vacua, and Gaussian states have a well developed theory [7, 8, 9, 10]. The present construction starts from a different end of that circle. The Gaussian state is first selected by a current constraint, and its coefficient recurrence then determines the kernel K_a. Only after this recurrence has been fixed do we construct the quadratic creator and prove that its closed powers reproduce the already selected sectors.

This ordering makes the analytic realization part of the result. Functional-integral and constructive treatments of quantum fields likewise place Hilbert-space completion and operator control alongside the formal expressions themselves [11, 12]. In the present setting the expression $\frac{1}{2} a^\dagger K_a a^\dagger$ does not by itself determine a closed operator; one must identify its graph closure, maximal coefficient domain, and adjoint relation [13, 14]. The coefficientwise exponential addresses a related but distinct issue: its coefficients are locally finite in occupation space, while its domain is fixed globally by square summability of the assembled output. The broader formalization program in quantum field theory makes this separation between symbolic formula and completed analytic statement especially explicit [15].

The resulting data may be summarized by $(K_a, \overline{A}_a^\dagger, \Xi_a, \Psi_a)$. The seed Ξ_a recovers the finite modulus and pair kernel, whereas Ψ_a carries the all-orders current constraint. Completed multiparticle carriers, discrete celestial bases, and asymptotic symmetry algebras provide a natural downstream setting in which such distinctions become representation-theoretic [3, 2, 6, 5, 4]. None of those asymptotic structures is assumed here. The next paper asks what the generated pair represents once Bose symmetry, spacetime symmetry, and particle semantics are introduced.

2 From precision to pair geometry

Let $\mathfrak{h}$ be the finite one-particle Hilbert space of the regulated model, with dimension d; in the formal instantiation used below, $d = 60$. The positive precision matrix G_a and amplitude covariance Σ_a satisfy

$$\Sigma_a(I + G_a) = 2I, \qquad \Sigma_a = I + K_a. \tag{13}$$

Since G_a is positive definite, $I + G_a$ is invertible, and

$$K_a = (I - G_a)(I + G_a)^{-1}, \qquad G_a = (I - K_a)(I + K_a)^{-1}. \tag{14}$$

If $G_a v = gv$ with $g > 0$, then

$$K_a v = \frac{1-g}{1+g}\, v, \qquad \left|\frac{1-g}{1+g}\right| < 1. \tag{15}$$

Thus the one-particle pair operator

$$(\mathsf{K}_a x)_r = \sum_s K_a(r, s) x_s \tag{16}$$

is self-adjoint and strictly contractive,

$$\mathsf{K}_a^* = \mathsf{K}_a, \qquad \|\mathsf{K}_a\| < 1. \tag{17}$$

Finite dimensionality makes K_a automatically Hilbert–Schmidt.

The three scale parameters enter G_a through independently recoverable positive blocks. Equality $K_a = K_b$ implies $G_a = G_b$ by Eq. (14), and the block readout then gives $a = b$. Hence

$$(0, \infty)^3 \hookrightarrow \{\text{self-adjoint strict contractions on } \mathfrak{h}\}. \tag{18}$$

The specific sector coordinates used in the formal model are recorded in the theorem map; they are not needed for the operator argument in the main text.

3 The quadratic pair creator

Let $\mathcal{C} \subset \mathcal{F}$ denote the finite-occupation core of the bosonic Fock space over $\mathfrak{h}$. For creation and annihilation maps $a_r^\dagger, a_r$, define

$$A^\dagger_{a,\mathrm{core}} = \frac{1}{2}\sum_{r,s} K_a(r, s) a_r^\dagger a_s^\dagger, \qquad A_{a,\mathrm{core}} = \frac{1}{2}\sum_{r,s} K_a(r, s) a_s a_r. \tag{19}$$

The sums are finite because $\mathfrak{h}$ is finite dimensional. If $\mathcal{C}^{(N)}$ denotes exact particle degree N, then

$$A^\dagger_{a,\mathrm{core}} : \mathcal{C}^{(N)} \longrightarrow \mathcal{C}^{(N+2)}. \tag{20}$$

In particular,

$$(A^\dagger_{a,\mathrm{core}})^n \Omega_{\mathrm{core}} \in \mathcal{C}^{(2n)}.$$

Symmetry of K_a also gives

$$\left\langle \iota A^\dagger_{a,\mathrm{core}}\phi, \iota\chi \right\rangle = \langle \iota\phi, \iota A_{a,\mathrm{core}}\chi \rangle, \qquad \phi, \chi \in \mathcal{C}, \tag{21}$$

where $\iota : \mathcal{C} \to \mathcal{F}$ is the dense inclusion. This pairing proves that the pair creator is closable.

3.1 The closed realization

Let

$$\overline{A}^\dagger_a := \overline{A^\dagger_{a,\mathrm{core}}}$$

be the graph closure. The finite core is contained in its domain and

$$\overline{A}^\dagger_a \, \iota\phi = \iota A^\dagger_{a,\mathrm{core}}\phi. \tag{22}$$

The same quadratic formula defines a maximal coefficient action. For a Fock coefficient function ψ, each output occupation depends on finitely many predecessor occupations. Write this formal output as $A^\dagger_{a,\mathrm{formal}}\psi$. Define

$$\mathrm{Dom}(A^\dagger_{a,\max}) = \left\{ \psi \in \mathcal{F} : A^\dagger_{a,\mathrm{formal}}\psi \in \mathcal{F} \right\}. \tag{23}$$

Particle cutoffs satisfy

$$P_{\leq N}\psi \to \psi, \qquad A^\dagger_{a,\max} P_{\leq N}\psi = P_{\leq N+2} A^\dagger_{a,\max}\psi \to A^\dagger_{a,\max}\psi. \tag{24}$$

Thus the finite core is a graph core for the maximal coefficient action, and

$$\overline{A}^\dagger_a = A^\dagger_{a,\max}. \tag{25}$$

If $\psi \in \mathrm{Dom}((A_{a,\mathrm{core}})^*)$ and $\eta = (A_{a,\mathrm{core}})^*\psi$, testing the adjoint identity against occupation kets gives

$$(A^\dagger_{a,\mathrm{formal}}\psi)(m) = \eta(m) \qquad \forall m. \tag{26}$$

Since $\eta \in \mathcal{F}$, the formal output is square summable. Hence

$$\overline{A^\dagger_{a,\mathrm{core}}} = A^\dagger_{a,\max} = (A_{a,\mathrm{core}})^*. \tag{27}$$

The three constructions have the same domain and action.

4 Factorial powers and sector reconstruction

Set

$$\Phi_{a,0}^{\mathrm{core}} = \Omega_{\mathrm{core}}, \qquad \Phi_{a,n}^{\mathrm{core}} = (A_{a,\mathrm{core}}^{\dagger})^n \Omega_{\mathrm{core}}. \tag{28}$$

Every $\Phi_{a,n}^{\mathrm{core}}$ has exact particle degree $2n$, and

$$(\overline{A}_a^{\dagger})^n \Omega = \iota \Phi_{a,n}^{\mathrm{core}}. \tag{29}$$

The canonical commutation relations give

$$a_s \Phi_{a,n+1}^{\mathrm{core}} = (n+1) \sum_r K_a(s,r) a_r^{\dagger} \Phi_{a,n}^{\mathrm{core}}. \tag{30}$$

After factorial normalization,

$$F_{a,n}^{\mathrm{core}} := \frac{1}{n!} \Phi_{a,n}^{\mathrm{core}},$$

this becomes

$$a_s F_{a,n+1}^{\mathrm{core}} = \sum_r K_a(s,r) a_r^{\dagger} F_{a,n}^{\mathrm{core}}. \tag{31}$$

At the coefficient level this is the recurrence already obtained from the Gaussian moment identities.

The recurrence, support condition, and vacuum coefficient determine the completed coefficient function uniquely. With

$$V_a := \langle \Omega, \Psi_a \rangle > 0, \tag{32}$$

the factorial candidate has the same vacuum value, support, parity, and recurrence as Ψ_a. Therefore

$$P_{2n} \Psi_a = \frac{V_a}{n!} (\overline{A}_a^{\dagger})^n \Omega, \qquad P_{2n+1} \Psi_a = 0. \tag{33}$$

This is an operator identity at every particle number.

Taking norms gives

$$\|P_{2n} \Psi_a\|^2 = \frac{|V_a|^2}{(n!)^2} \left\| (\overline{A}_a^{\dagger})^n \Omega \right\|^2, \tag{34}$$

and normalization of Ψ_a yields

$$1 = |V_a|^2 \sum_{n=0}^{\infty} \frac{\|(\overline{A}_a^{\dagger})^n \Omega\|^2}{(n!)^2}. \tag{35}$$

The exact sectors are mutually orthogonal, so

$$\Psi_a = V_a \sum_{n=0}^{\infty} \frac{1}{n!} (\overline{A}_a^{\dagger})^n \Omega \tag{36}$$

with unconditional convergence in Fock norm.

5 The faithful two-particle seed

The sector formula makes the first nonvacuum component especially transparent. Since $V_a > 0$, we may remove the vacuum amplitude and define

$$\Xi_a := V_a^{-1} P_2 \Psi_a. \tag{37}$$

Setting $n = 1$ in Eq. (33) gives

$$\Xi_a = \overline{A}_a^{\dagger} \Omega. \tag{38}$$

Thus the normalized seed is simultaneously the degree-two component of the selected Gaussian and the vacuum image of the closed pair creator. It is nonzero throughout the positive family.

To see how much information this vector retains, let e_{rs} be the normalized two-particle occupation ket associated with the coordinates r and s, including the usual diagonal normalization when $r = s$. For a fixed positive normalization N_{rs},

$$\langle e_{rs}, P_2 \Psi_a \rangle = \frac{V_a \, K_a(r, s)}{N_{rs}}. \tag{39}$$

The two-particle coefficients therefore recover the entries of the pair kernel directly. After division by the nonzero vacuum coefficient, the normalized seed Ξ_a determines K_a entry by entry.

Combining this readout with the injectivity of the pair-kernel parametrization gives

$$\Xi_a = \Xi_b \quad \Longrightarrow \quad K_a = K_b \quad \Longrightarrow \quad a = b. \tag{40}$$

In particular,

$$(0, \infty)^3 \hookrightarrow P_2 \mathcal{F}, \qquad a \longmapsto \Xi_a, \tag{41}$$

is an injective realization of the Gaussian moduli inside the finite degree-two sector. This is a statement about identifiability, not truncation: Ξ_a labels the completed family faithfully, while the bounded-particle exclusion theorem still prevents any nonzero finite sector from replacing Ψ_a as an exact current-physical state.

The same distinction can be expressed through the overlap kernel of the completed family. Let

$$\mathfrak{P} := \{\Xi_a : a \in (0, \infty)^3\}$$

be the image of the seed map. Injectivity provides an inverse parameter map $a(\Xi)$ on $\mathfrak{P}$, and hence a kernel

$$\Gamma_{\mathrm{rec}}(\Xi, \Xi') := \big\langle \Psi_{a(\Xi)}, \Psi_{a(\Xi')} \big\rangle. \tag{42}$$

For points in the seed image,

$$\Gamma_{\mathrm{rec}}(\Xi_a, \Xi_b) = \langle \Psi_a, \Psi_b \rangle. \tag{43}$$

The construction recovers the parameters and therefore the completed-state overlap; it does not identify the Gram matrix of the pair seeds with the Gram matrix of the completed Gaussian states. The two geometries encode different levels of the same family.

6 The coefficientwise exponential

For $\psi \in \mathcal{F}$, define at each occupation m

$$(\mathcal{E}_a^{\text{form}}\psi)(m) := \sum_{n=0}^{\lfloor N(m)/2 \rfloor} \frac{1}{n!} \Big((A^\dagger_{a,\text{formal}})^n \psi \Big)(m). \tag{44}$$

Only finitely many terms contribute to a fixed output occupation. Set

$$\mathcal{D}_{\exp}(a) := \Big\{ \psi \in \mathcal{F} : \mathcal{E}_a^{\text{form}}\psi \in \mathcal{F} \Big\}, \qquad \mathcal{E}_a\psi := \mathcal{E}_a^{\text{form}}\psi. \tag{45}$$

This is the maximal square-summable realization of the coefficientwise exponential.

For each occupation, coefficient evaluation is continuous and Eq. (44) is a finite linear combination of predecessor evaluations. Hence

$$\Gamma(\mathcal{E}_a) = \bigcap_m \Big\{ (\psi, \eta) : (\mathcal{E}_a^{\text{form}}\psi)(m) = \eta(m) \Big\} \tag{46}$$

is closed. The vacuum belongs to $\mathcal{D}_{\exp}(a)$, and

$$\mathcal{E}_a\Omega = \sum_{n=0}^{\infty} \frac{1}{n!} (\overline{A}_a^\dagger)^n \Omega, \qquad \Psi_a = V_a\, \mathcal{E}_a \Omega. \tag{47}$$

The finite-occupation core is contained in $\mathcal{D}_{\exp}(a)$ and is dense in $\mathcal{F}$. On that core the coefficientwise definition agrees with the norm-convergent series of closed-creator iterates.

7 Spectral product and determinant normalization

Because K_a is self-adjoint on a finite-dimensional one-particle space, choose an orthonormal eigenbasis with

$$K_a u_j = \kappa_j u_j, \qquad j = 1, \ldots, d.$$

Second quantization of the diagonalizing unitary preserves the vacuum and carries the pair creator to

$$\frac{1}{2} \sum_{j=1}^{d} \kappa_j (b_j^\dagger)^2 \tag{48}$$

on the finite-occupation core.

For one spectral channel,

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n} (r\kappa)^{2n} = \frac{1}{\sqrt{1 - r^2\kappa^2}}, \qquad |r\kappa| < 1. \tag{49}$$

Finite Cauchy products over the spectral channels give

$$\prod_{j=1}^{d} \frac{1}{\sqrt{1 - r^2\kappa_j^2}} = \det(I - r^2 K_a^* K_a)^{-1/2}. \tag{50}$$

The same counting extends from the vacuum to every finite occupation, which gives summability of the coefficientwise exponential on the full finite core.

For

$$E_a(t) := \sum_{n=0}^{\infty} \frac{t^n}{n!} (\overline{A}_a^{\dagger})^n \Omega, \tag{51}$$

orthogonality of the $2n$-particle sectors gives

$$\|E_a(t)\|^2 = \sum_{n=0}^{\infty} \frac{|t|^{2n}}{(n!)^2} \left\|(\overline{A}_a^{\dagger})^n \Omega\right\|^2 = \det\Big(I - |t|^2 K_a^* K_a\Big)^{-1/2} \tag{52}$$

for

$$|t|\,\|K_a\| < 1.$$

No converse convergence classification outside this region is needed.

The Gaussian integral fixes the vacuum coefficient:

$$V_a^4 = \det(I - K_a^* K_a) > 0. \tag{53}$$

With the positive phase convention, V_a is the positive fourth root. At $t = 1$,

$$|V_a|^2 \|E_a(1)\|^2 = 1. \tag{54}$$

The same determinant therefore controls the norm of the unnormalized pair series and the prefactor which normalizes the state.

8 Completion and the current constraint

Let

$$S_N(a) := V_a \sum_{n=0}^{N} \frac{1}{n!} (\overline{A}_a^{\dagger})^n \Omega. \tag{55}$$

Every $S_N(a)$ is nonzero and has bounded particle support. The bounded-particle exclusion theorem from the companion analysis and the norm convergence of Eq. (36) imply

$$S_N(a) \notin \mathcal{K}_{\text{joint}} \quad \forall N, \qquad S_N(a) \longrightarrow \Psi_a \in \mathcal{K}_{\text{joint}}. \tag{56}$$

No finite truncation is an exact current-physical state, although the sequence converges to one.

Successive sectors are tied by the same recurrence, so Eq. (56) identifies the completed state as the closure of a single cross-sector law. In this family the state is finitely governed but nontruncatably realized.

9 Transport to a completed momentum carrier

The regulated Fock representation admits an isometric embedding

$$\iota : \mathcal{F} \longrightarrow \mathcal{F}_{\text{cont}}$$

into a completed momentum carrier. Let

$$\mathcal{H}^{\mathrm{im}} := \iota(\mathcal{F})$$

be its closed image. Conjugating the closed pair creator and the coefficientwise exponential through the unitary identification $\mathcal{F} \simeq \mathcal{H}^{\mathrm{im}}$ gives

$$\begin{aligned} \overline{A}^{\dagger}_{a,\mathrm{im}}\, \iota\psi &= \iota\, \overline{A}^{\dagger}_{a}\psi, \\ (\overline{A}^{\dagger}_{a,\mathrm{im}})^{n}\Omega_{\mathrm{im}} &= \iota(\overline{A}^{\dagger}_{a})^{n}\Omega, \\ \mathcal{E}_{a,\mathrm{im}}\Omega_{\mathrm{im}} &= \iota\, \mathcal{E}_{a}\Omega. \end{aligned} \tag{57}$$

Consequently,

$$\Psi^{\mathrm{cont}}_{a} = V_a\, \mathcal{E}_{a,\mathrm{im}}\Omega_{\mathrm{im}}. \tag{58}$$

This transport is exact on the closed image of the regulated representation. It does not construct a pair creator on the whole continuum carrier and does not remove the regulator; those are separate questions.

10 Discussion

The current constraint and the pair recurrence together produce a rather economical description of the Gaussian family. The recurrence fixes K_a, the Cayley transform relates it to the positive precision G_a, and the closed quadratic creator generated by K_a reproduces every even sector. The vacuum determinant fixes the overall normalization. What initially appears as an infinite collection of Fock coefficients is therefore organized by a finite one-particle kernel and a single closed operator.

The normalized two-particle seed sharpens this picture. Its coefficient functions recover K_a, and equality of seeds forces equality of the Gaussian parameters. The completed state is therefore identifiable from finite-degree data even though it is not itself a finite-degree state. This is the sense in which finite generation and nontruncatable realization coexist here: one should distinguish the information needed to specify a point in the family from the number of sectors required for that point to satisfy the exact current constraint.

The operator-theoretic statements are essential to this conclusion. The formal quadratic expression

$$\frac{1}{2} a^{\dagger} K_a a^{\dagger}$$

becomes a definite unbounded operator only after its domain is fixed, and Eq. (27) shows that the graph closure, maximal coefficient action, and Hilbert adjoint select the same realization. The coefficientwise exponential solves a different problem: its output is locally finite coefficient by coefficient, while membership in its domain is determined by a global square-summability condition. On the finite core this operator agrees with the norm-convergent power series of the closed creator.

Two scope restrictions will matter later. The determinant identity for $E_a(t)$ is established on $|t|\,\|K_a\| < 1$; no converse convergence classification outside this region is needed here. Likewise, the momentum-space transport above is an exact change of carrier at the fixed regulator, rather than a continuum limit. Finally, recovering the completed overlap kernel from Ξ_a uses the injective parameter map and does not equate the pair-seed Gram matrix with the completed-state Gram matrix. With these distinctions in place, the next question is naturally representation theoretic: what does the faithful degree-two seed represent once Bose symmetry, spacetime symmetry, and particle semantics are introduced?

11 Conclusion

The Gaussian recurrence determines a symmetric pair kernel and a canonical closed quadratic creator satisfying

$$\overline{A^\dagger_{a,\mathrm{core}}} = A^\dagger_{a,\max} = (A_{a,\mathrm{core}})^*, \tag{59}$$

and its factorial powers reproduce the exact sectors,

$$P_{2n}\Psi_a = \frac{V_a}{n!}(\overline{A}^\dagger_a)^n\Omega, \qquad P_{2n+1}\Psi_a = 0. \tag{60}$$

The same reconstruction is encoded by the closed coefficientwise exponential,

$$\Psi_a = V_a\,\mathcal{E}_a\Omega. \tag{61}$$

At degree two,

$$\Xi_a := V_a^{-1}P_2\Psi_a = \overline{A}^\dagger_a\Omega \tag{62}$$

is faithful to the Gaussian modulus,

$$\Xi_a = \Xi_b \implies K_a = K_b \implies a = b. \tag{63}$$

The same kernel controls the spectral norm and vacuum normalization,

$$\|E_a(t)\|^2 = \det(I - |t|^2K_a^*K_a)^{-1/2}, \qquad V_a^4 = \det(I - K_a^*K_a).$$

Yet the finite partial sums satisfy

$$S_N(a) \notin \mathcal{K}_{\mathrm{joint}} \quad \forall N, \qquad S_N(a) \longrightarrow \Psi_a \in \mathcal{K}_{\mathrm{joint}}. \tag{64}$$

Thus a finite pair kernel both generates the completed state and identifies its modulus, while exact current physicality belongs to the completed hierarchy rather than to any nonzero finite truncation.

A Lean theorem map

The main text uses representation-theoretic notation and reserves project-specific carrier and sector names for this theorem map. The table below gives the exact Lean counterparts. The core pair-generation chain remains CI139–CI170; the revised manuscript also uses later exact corollaries identifying the faithful two-particle seed. The combined manuscript surface is audited against repository commit

`1206331994899b5ecffb2297c4a18838ee4fd0b2`.

A.1 Core pair-generation chain

Mathematical result	Formal module
Positive-orthant pair-kernel compatibility	`CI139.CanonicalPositiveGaussianPairKernelCompatibility`
Precision/pair Cayley transform	`CI140.CanonicalPositiveGaussianCayleyGeometry`
Finite pair-block geometry	`CI141.CanonicalPositiveGaussianPairBlockGeometry`
Self-adjoint contractive pair operator	`CI142.CanonicalPositiveGaussianPairKernelOperator`
Dense-core creator and annihilator	`CI143.CanonicalPositiveGaussianPairCoreOperators`
Pair powers and factorial recurrence	`CI144.CanonicalPositiveGaussianPairPowers`
Recurrence uniqueness	`CI145.CanonicalPositiveGaussianPairRecurrenceUniqueness`
Factorial-series reconstruction	`CI146.CanonicalPositiveGaussianPairSeriesReconstruction`
Canonical closed pair creator	`CI147.CanonicalPositiveGaussianClosedPairCreator`
Exact sectors and vacuum pair series	`CI148.CanonicalPositiveGaussianClosedSectorExponential`
Coefficient-maximal exponential	`CI149.CanonicalPositiveGaussianMaximalExponential`
Closed graph of the maximal exponential	`CI150.CanonicalPositiveGaussianMaximalExponentialClosed`
Finite truncation/current-kernel theorem	`CI151.CanonicalPositiveGaussianFiniteResummationPhysicality`
Determinant normalization	`CI152.CanonicalPositiveGaussianDeterminantNormalization`
Scale/kernel/generator injectivity	`CI153.CanonicalPositiveGaussianGeneratorInjectivity`
Exact transport to a closed continuum-momentum image	`CI154.CanonicalPositiveGaussianR3OperatorTransport`
Sector Parseval normalization	`CI155.CanonicalPositiveGaussianSectorNormalization`
Particle-cutoff graph core	`CI156.CanonicalPositiveGaussianCoordinateMaximalCreator`
Maximal Hilbert-adjoint identification	`CI157.CanonicalPositiveGaussianMaximalAdjoint`
Complete pair-generation surface	`CI158.CanonicalPositiveGaussianPairGenerationSurface`
Spectral product	`CI161.CanonicalTowerNativeSqueezedVacuumSpectralProduct`
Fock-space spectral diagonalization	`CI164.CanonicalPositiveGaussianSpectralFockBridge`
Diagonal occupation counting	`CI165.CanonicalPositiveGaussianDiagonalOccupationCount`
Operator-norm spectral-disk determinant law	`CI166.CanonicalPositiveGaussianSharpVacuumExponential`
Finite-occupation spectral series	`CI168.CanonicalPositiveGaussianFiniteOccupationSpectralSeries`
Dense exponential domain	`CI169.CanonicalPositiveGaussianFiniteCoreExponentialDomain`
Terminal core pair-generation surface	`CI170.CanonicalPositiveGaussianSharpDensePairGenerationSurface`
Input positive current-constrained family	`CI134.CanonicalPositiveOrthantGaussianPhysicalFamily`
All-orders pair recurrence	`CI136.CanonicalPositiveOrthantGaussianPairRecurrence`

A.2 Later faithful-seed corollaries

Mathematical result	Formal module
Exact degree-two pair-seed carrier	`Tier30.Completion.CanonicalGaussianPairSeedCarrier`
Pair seed equals pair creator on vacuum	`Tier30.Completion.CanonicalPairSeedPairCreatorJunction`
Normalized pair-seed injectivity	`Tier30.Completion.CanonicalPairSeedInjectivityDecision`
Canonical pair-seed coefficient readout	`Tier30.Completion.CanonicalPairSeedCoefficientReadout`
Recovered completed-state kernel on the seed image	`Tier30.Completion.CanonicalPairSeedSchurGeometry`

Internal names such as `Tower`, `Witness`, `R3`, and sector-specific gauge labels record formal provenance rather than manuscript ontology. The manuscript suppresses those names where they do not change the mathematical statement, and their presence in a module name adds no semantic authority beyond the theorem stated in the main text. Likewise, internal module names containing `Sharp` refer to the proved operator-norm spectral region; no converse convergence theorem outside $|t|\,\|K_a\| < 1$ is asserted.

B Formal development and collaboration

The results are supported by a Lean 4 formal development maintained in a private research repository. The Paper 2 theorem surface, including the terminal CI139–CI170 pair-generation chain and the later faithful two-particle seed corollaries, is audited against repository commit

`1206331994899b5ecffb2297c4a18838ee4fd0b2`.

The audited environment uses Lean 4 `v4.28.0-rc1` and mathlib revision `5352afccd6866369be9de43f5b7ec47203555f44`. The theorem map above provides the correspondence between the manuscript notation and the project declarations.

The full repository is not publicly accessible. Paper-scoped source review, theorem-level verification, and access to the relevant formal modules may be arranged in the context of professional collaboration, technical review, or institutional partnership.

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