Source-Aware Weyl-Reduced Celestial Representation

Joint source-output symmetry and positive-measure obstruction to fixed-source isometry for radiative two-particle Clebsch–Gordan kernels

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Abstract

The preceding paper constructs a qualified physical radiative Bose pair and hands forward its generic completed two-particle Hilbert space. The present paper studies the Weyl/shadow structure of the normalized celestial Clebsch–Gordan kernel at fixed external representation labels and tests whether its contact-normalized transition can be realized fiberwise by isometries.

For fixed external labels, the normalized kernel determines the source unitary $U^{\mathrm{src}}_{J,\nu}$, the unit-modulus channel phase $\gamma_{J,\nu}$ and the completed angular shadow $\mathcal{S}_{-\nu,J}$. These data generate a bounded source-aware operator-family class with rows

$$\mathcal{C}_{J,\nu} : \mathcal{H}_{\mathrm{src}} \longrightarrow \mathcal{H}^{\mathrm{flat}}$$

satisfying

$$\mathcal{C}_{J,\nu}[\psi] = \gamma_{J,\nu}\, \mathcal{S}_{-\nu,J}\mathcal{C}_{-J,-\nu}[U^{\mathrm{src}}_{J,\nu}\psi].$$

The physical Clebsch–Gordan kernel simultaneously defines dense-core angular coefficients whose completed direct-integral assembly is treated in Paper 6. The fixed-external-label source Hilbert space plays the role of a multiplicity space in the joint source/output spectral representation. The source unitaries form a strongly continuous representation of $\mathbb{Z} \times \mathbb{R}$. In source spectral coordinates they act by

$$(z, \omega, k) \longmapsto \left(z, \omega + \frac{\nu}{\pi}, k + 2J\right).$$

Combined with principal-series Weyl reflection, this action defines a measure-preserving involution on the joint principal-series/source spectral space. Its fixed subspace is unitarily equivalent, by $\sqrt{2}$-normalized restriction, to the L^2 space over a fundamental domain of the Weyl reflection, with the full source Hilbert space retained. The reflected row is paired with the transformed source $U^{\mathrm{src}}_{J,\nu}\psi$, so the $\sqrt{2}$ factor is a norm identity of this joint Weyl-paired representation. We refer to this construction as the source-Weyl representation.

The contact-normalized Clebsch–Gordan transition coefficient obeys a stronger metric constraint under any fiberwise isometric realization. An independent reconstruction from the original physical kernel gives

$$\mathcal{T}_{1,0} = 1 - 4\log 2$$

on the equal-positive-helicity spin-one row. Joint continuity in the two incoming Mellin frequencies and the output principal-series frequency extends the inequality $|\mathcal{T}|^2 > 1$ to an open subset of $\mathbb{R}^3$ with positive joint external/output spectral measure. Any fiberwise isometric realization on normalized source and target vectors requires $|\mathcal{T}|^2 = 1$. Hence no almost-everywhere fixed-source isometric realization agrees with the normalized physical kernel.

The output-shadow distributional completion admits an exact classification. Every tempered extension in the output-shadow angular variable that agrees with the spin-one punctured kernel differs by a unique Dirac term,

$$D_c = D_{\mathrm{can}} + c\,\delta_0,$$

and the normalized transition shifts as $\mathcal{T}_c = \mathcal{T}_{\mathrm{can}} + c$. Unrestricted contact freedom can restore unit modulus. The homogeneous shadow-extension law selects $c = 0$, in agreement with the canonical Fourier/contact normalization, so the homogeneous/Fourier-normalized completion retains the Gram defect.

The source action has an equally precise change-of-trivialization classification. Scalar row phases preserve the Gram factor. Every label-independent unitary source-basis change preserves the nontriviality of the spin-one source action up to conjugacy. An explicit label-dependent measurable source-frame gauge trivializes the bare two-sheet cocycle. Here “gauge” refers only to unitary changes of source-space trivialization and is unrelated to spacetime gauge symmetry.

The paper therefore separates the Weyl-fixed joint spectral representation from the angular-valued kernel range, classifies the exact normalization freedom of the singular kernel, and identifies

the positive-measure obstruction to fixed-source isometry. These results provide the representation-theoretic input for the physical angular Clebsch–Gordan transform and range analysis developed in Paper 6.

1 Introduction

The first four papers in this series proceed from constrained completion to a physically qualified radiative pair. Paper 1 constructs a nontruncatable completed current-zero sector with an injective Gaussian family [1]. Paper 2 identifies the faithful quadratic pair geometry that generates its even Fock tower [2]. Paper 3 establishes the Bose/Fock, analytic and Poincaré structure of the generated degree-two sector and records the boundary of direct asymptotic-particle identification [3]. Paper 4 studies the dynamical descendants of that Gaussian geometry and constructs a qualified physical radiative Bose pair [4].

Paper 4 ends with the generic handoff

$$\Phi^{(2)} \in \mathcal{H}^{(2)}_{\text{phys}}. \tag{1}$$

The present paper begins from this completed two-particle setting. Writing $\mathcal{D}_{\text{pair}} \subset \mathcal{H}^{(2)}_{\text{phys}}$ for the dense pair core used by the kernel, the fixed-external-label realization is

$$\boxed{\Psi \in \mathcal{D}_{\text{pair}} \xmapsto{\iota_{a,b}} \psi_{a,b} := \iota_{a,b}\Psi \in \mathcal{H}_{\text{src}}.} \tag{2}$$

The source vector is therefore the fixed-external-label realization of the same two-particle input used in the kernel calculation.

For fixed external representation labels, the two-particle kernel acts on a Hilbert space of source wavefunctions. This space plays the role of a multiplicity space in the joint source/output spectral representation, and we call it the *source Hilbert space* $\mathcal{H}_{\text{src}}$. The central questions are

> *How does principal-series Weyl/shadow reflection act jointly on the output label and this source Hilbert space?*
>
> *Can the normalized Clebsch–Gordan transition be realized fiberwise by isometries on the joint external/output spectral parameter space?*

These questions define two mathematical lines:

$$\boxed{\begin{array}{c}\text{joint source/output principal-series representation}\\ \downarrow \\ \text{Weyl-invariant subspace}\\ \downarrow \\ \text{restriction to a fundamental domain}\\[6pt] \text{and independently}\\[6pt] \text{normalized physical Clebsch–Gordan kernel}\\ \downarrow \\ \text{fixed-source isometry test.}\end{array}} \tag{3}$$

For fixed external labels, let

$$\mathcal{C}_{J,\nu} : \mathcal{H}_{\text{src}} \longrightarrow \mathcal{H}^{\text{flat}} \tag{4}$$

denote a row of the bounded source-aware operator-family class generated by the kernel-derived source and shadow data. Its completed source-twisted relation is

$$\boxed{\mathcal{C}_{J,\nu}[\psi] = \gamma_{J,\nu}\, \mathcal{S}_{-\nu,J}\mathcal{C}_{-J,-\nu}[U^{\mathrm{src}}_{J,\nu}\psi].} \tag{5}$$

Here $\gamma_{J,\nu}$ has unit modulus, $\mathcal{S}_{-\nu,J}$ is the normalized integral-spin shadow intertwiner and $U^{\mathrm{src}}_{J,\nu}$ is a unitary on the source Hilbert space. The literal angular Clebsch–Gordan coefficient on the dense kernel core is denoted separately by $C^{a,b,\mathrm{core}}_{J,\nu}[\Psi]$ below; its completed direct-integral range is a downstream analytic object.

The source unitaries form an additive representation. Writing $g = (J,\nu) \in \mathbb{Z}\times\mathbb{R}$,

$$\boxed{U_0 = I, \qquad U_gU_h = U_{g+h}, \qquad U_{-g} = U_g^{-1}.} \tag{6}$$

In the Rühl/log-cylinder coordinates of Section 2, their character is

$$\chi_{J,\nu}(t,\theta) = e^{-2i\nu t}e^{-2iJ\theta}, \tag{7}$$

and the spectral representation of the source space gives

$$\boxed{\Theta_{J,\nu}(z,\omega,k) = \left(z, \omega + \frac{\nu}{\pi}, k + 2J\right).} \tag{8}$$

The output reflection and source translation combine into the measure-preserving involution

$$\boxed{\mathscr{W}(J,\nu,\xi) = (-J,-\nu,\Theta_{J,\nu}\xi).} \tag{9}$$

Because the shadow intertwiner acts simultaneously on the principal-series label and on the source Hilbert space, we refer to Eqs. (5)–(9) as the *source-aware shadow relation* and the *source-Weyl action*.

The representation-theoretic discovery is the kernel-derived source action in Eqs. (5)–(9). Its Hilbert-space consequence is the exact Weyl reduction

$$\boxed{\mathcal{A} : \mathcal{H}^{\mathrm{fix}}_{\mathrm{joint}} \xrightarrow{\simeq_u} \mathcal{H}_{\mathrm{red}}, \qquad \mathcal{A}F = \sqrt{2}\, F|_{\mathcal{F}}.} \tag{10}$$

Here $\mathcal{H}^{\mathrm{fix}}_{\mathrm{joint}} = \{F : \mathsf{W}F = F\}$ is the Weyl-fixed joint source/output spectral subspace and

$$\mathcal{H}_{\mathrm{red}} = L^2(\mathcal{F}\times X_{\mathrm{src}})$$

is the space over a fundamental domain $\mathcal{F}$ of the output Weyl reflection, with the entire source Hilbert space retained. Analysis, synthesis, Parseval, both inverse identities and measurable external phase assembly are exact.

The physical pair in Eq. (1), the Weyl-fixed joint L^2 space and the angular-vector-valued Clebsch–Gordan coefficient space remain mathematically distinct. The scalar fixed subspace records the representation geometry implied by the exact kernel symmetry. The angular transform and its range are separate analytic objects.

The normalized transition coefficient provides the metric result. Let

$$\mathcal{T}^{a,b}_{J,\nu} \tag{11}$$

denote the contact-normalized scalar transition for fixed external labels. If an isometry V realizes this coefficient on unit source and target vectors,

$$V e_s = \mathcal{T}^{a,b}_{J,\nu} e_t, \qquad \|e_s\| = \|e_t\| = 1,$$

then

$$\boxed{|\mathcal{T}^{a,b}_{J,\nu}|^2 = 1.} \tag{12}$$

For equal positive external helicities at zero incoming Mellin frequencies and output label $(J,\nu) = (1,0)$, an independent reconstruction from the normalized physical kernel gives

$$\boxed{\mathcal{T}_{1,0} = 1 - 4\log 2.} \tag{13}$$

The reconstruction derives the Plancherel normalization, normalized spherical state, shadow factor, contact coefficient, contour residues, moving-pole chambers and finite-part plane functional before identifying the unevaluated correction with the completed transition.

The decisive measure-theoretic result varies both incoming Mellin frequencies and the output principal-series frequency. Define

$$\mathcal{T}_3(\nu_1, \nu_2, \nu)$$

on the equal-positive-helicity spin-one family. Joint continuity at $(0,0,0)$ and Eq. (13) give an open ball $B_\epsilon(0) \subset \mathbb{R}^3$ such that

$$\boxed{|\mathcal{T}_3(\nu_1,\nu_2,\nu)|^2 > 1 \qquad \text{for } (\nu_1,\nu_2,\nu) \in B_\epsilon(0).} \tag{14}$$

This ball has positive measure with respect to the joint external/output spectral measure. Equation (12) therefore yields

$$\boxed{\begin{array}{c}\text{no almost-everywhere fixed-source isometric realization}\\ \text{agrees with the normalized physical kernel.}\end{array}} \tag{15}$$

The singular extension has an exact normalization classification. On the audited spin-one row, every tempered extension agreeing with the punctured kernel has the form

$$\boxed{D_c = D_{\text{can}} + c\,\delta_0, \qquad \mathcal{T}_c = \mathcal{T}_{\text{can}} + c.} \tag{16}$$

Unrestricted c permits unit-modulus repairs. Compatibility with the homogeneous shadow-extension law requires

$$D_c(\mathcal{D}_\lambda \varphi) = m_{0,1}(\lambda)\, D_c(\varphi) \qquad (\lambda \in \mathbb{C}^\times), \tag{17}$$

where $\mathcal{D}_\lambda$ is the test-function dilation and $m_{0,1}$ is the spin-one shadow dilation character. This homogeneity law holds exactly when

$$\boxed{c = 0.} \tag{18}$$

The canonical Fourier/contact normalization selects the same extension.

Finally, the source action admits a precise change-of-trivialization classification. Scalar row rephasings preserve $|\mathcal{T}|^2$. Label-independent unitary changes of source basis conjugate U_g and preserve the nontriviality of the spin-one source action. An explicit label-dependent measurable source-frame gauge trivializes the bare two-sheet cocycle. Here "gauge" refers solely to source-space trivialization and is unrelated to spacetime gauge symmetry.

The paper is organized accordingly. The first part constructs the joint source/output representation and its Weyl reduction. The second identifies the normalized transition and its spin-one value, establishes the positive-measure isometry obstruction, and classifies completion and source-frame freedom. The closing sections give the interpretation and series handoff. Historical retained-sheet conditions are collected in the appendix as provenance.

1.1 Scope

The manuscript establishes a global Weyl-fixed joint source/output representation and a positive-measure obstruction to fixed-source isometry for the normalized two-particle Clebsch–Gordan transition. It also classifies the finite point-supported extension freedom and the unitary source-frame freedom relevant to these statements. The analysis remains at the two-particle kernel and global representation level. Coincident-point local OPEs, interacting collinear coefficients, soft-current algebras, detector tomography and faithful boundary reconstruction belong to later stages of the series.

1.2 Relation to nearby work

The unitary representation theory of the Lorentz group organizes massless one-particle states by little-group data and admits principal-series harmonic analysis [5, 6, 7]. In celestial amplitudes, Mellin transforms reorganize momentum-space states into conformal-primary bases on the celestial sphere [9, 10, 11]. Multiparticle celestial representations naturally introduce tensor-product decomposition, Clebsch–Gordan kernels and composite channels [12].

The reflection $(J,\nu) \mapsto (-J,-\nu)$ belongs to the harmonic-analytic setting of Knapp–Stein intertwiners [8]. Here the normalized integral-spin shadow acts together with a unitary transformation of the fixed-external-label source space. After this standard structure is established, we use the short term *source-Weyl action* for the combined output reflection and induced source-space action.

A direct-integral Hilbert space can encode the Weyl symmetry of a transform without coinciding with the transform's vector-valued range. This paper therefore distinguishes the Weyl-fixed joint spectral subspace from the angular Hilbert space in which the Clebsch–Gordan coefficients take values.

1.3 Dictionary to standard celestial conventions

The output principal-series labels used below translate directly to the standard celestial notation. We write

$$\boxed{\Delta = 1 + i\nu, \qquad h = \frac{\Delta + J}{2} = \frac{1 + i\nu + J}{2}, \qquad \bar{h} = \frac{\Delta - J}{2} = \frac{1 + i\nu - J}{2}.} \tag{19}$$

Thus the reflection used in this paper,

$$(J,\nu) \longmapsto (-J,-\nu), \tag{20}$$

is the usual two-dimensional shadow transformation

$$(\Delta, J) \longmapsto (2-\Delta, -J), \qquad (h, \bar{h}) \longmapsto (1-h, 1-\bar{h}), \tag{21}$$

restricted to the principal continuous series [10, 11].

An external one-particle label a consists of a graviton helicity and a real Fourier frequency f_a. Its celestial spin is

$$j_a \in \{+2, -2\}, \tag{22}$$

and the principal-series/Mellin frequency used in the conformal dimension is

$$\boxed{\nu_a = 2\pi f_a, \qquad \Delta_a = 1 + i\nu_a, \qquad h_a = \frac{1 + i\nu_a + j_a}{2}, \qquad \bar{h}_a = \frac{1 + i\nu_a - j_a}{2}.} \tag{23}$$

The output frequency ν used throughout the main calculation is the principal-series frequency itself.

Three real frequency variables appear in the paper and play distinct roles:

Symbol	Origin	Role
f_a	external one-particle Fourier coordinate	converted to the external Mellin frequency $\nu_a = 2\pi f_a$
ν	composite output label	principal-series parameter in $\Delta = 1 + i\nu$
ω	source log-radius Fourier dual	continuous coordinate in the source spectrum X_{src}

Thus the shift $\omega \mapsto \omega + \nu/\pi$ derived later acts on the source spectrum and uses the composite output frequency ν.

For two external labels a, b, the dense-core row $C^{a,b,\mathrm{core}}_{J,\nu}$ is the Clebsch–Gordan coefficient associated with the output principal-series representation $(\Delta, J) = (1 + i\nu, J)$. In conventional multiparticle language it resolves the tensor product of the two external principal-series states into a composite channel labelled by (Δ, J) [12]. The coefficient is angular-vector-valued. The scalar construction studied below is the Weyl-invariant joint source/output spectral subspace and its restriction to a fundamental output domain. The dictionary aligns the representation labels and keeps these Hilbert spaces distinct.

This paper	Conventional celestial notation	Meaning
(J, ν)	$(J, \Delta = 1 + i\nu)$	output principal-series channel
$(-J, -\nu)$	$(-J, 2 - \Delta)$	shadow-reflected output channel
$j_a = \pm 2$	spin/helicity $J_a = \pm 2$	external graviton celestial spin
f_a	Fourier variable	pre-Mellin frequency coordinate
$\nu_a = 2\pi f_a$	Mellin/principal-series frequency	$\Delta_a = 1 + i\nu_a$
$C^{a,b,\mathrm{core}}_{J,\nu}$	dense-core multiparticle CG row	angular coefficient in the (Δ, J) channel
(z, ω, k)	source spectral variables	retained affine source coordinate, log-radius frequency and angular Fourier mode
$\mathcal{H}_{\mathrm{red}}$	—	scalar L^2 space on one output Weyl fundamental domain with the full source space retained

1.4 Plancherel density and flat-frequency convention

For fixed integral spin J, the pulled-back Naimark Plancherel density in the conventions used here is

$$\boxed{\rho_{\mathrm{Pl}}(J,\nu) = 8(J^2+\nu^2).} \tag{24}$$

Thus the physical fixed-spin coefficient space and its flat-frequency realization are related by the exact unitary

$$\boxed{\mathcal{U}_J : L^2(\mathbb{R}, \rho_{\mathrm{Pl}}(J,\nu)\,d\nu) \xrightarrow{\simeq_u} L^2(\mathbb{R}, d\nu), \qquad (\mathcal{U}_J f)(\nu) = \sqrt{8(J^2+\nu^2)}\,f(\nu).} \tag{25}$$

The inverse multiplies almost everywhere by $[8(J^2+\nu^2)]^{-1/2}$. In the zero-spin row the sole zero occurs at $\nu = 0$, a Lebesgue-null point, so the completed inverse is exact at the L^2 level. The flat $d\nu$ carriers used below therefore retain the full weighted Plancherel norm through Eq. (25). At the spin-one audit point,

$$\rho_{\mathrm{Pl}}(1,0) = 8, \qquad \sqrt{\rho_{\mathrm{Pl}}(1,0)} = 2\sqrt{2}. \tag{26}$$

The angular Fourier period is 2π.

2 Physical pair input and representation spaces

This section fixes all spaces needed later so that the remainder of the paper can be read independently of Papers 1–4.

2.1 Radiative one- and two-particle spaces

The physical one-particle radiative Hilbert space is the two-helicity direct sum

$$\mathcal{H}^{(1)}_{\mathrm{rad}} = \bigoplus_{\lambda\in\{+2,-2\}} L^2(\mathbb{R}^3, d^3p), \tag{27}$$

with the established massless Wigner/Poincaré action on the momentum variable and helicity labels. Equation (27) uses the ordinary-volume momentum chart adopted by the formalization. The invariant positive-energy null-cone measure is related to this chart by an explicit density-normalization unitary, so the use of d^3p fixes a Hilbert-space coordinate convention for the same completed massless representation.

Let

$$\mathcal{H}_{\mathrm{pair}} = \overline{\mathcal{H}^{(1)}_{\mathrm{rad}} \otimes \mathcal{H}^{(1)}_{\mathrm{rad}}} \tag{28}$$

be the completed tensor square and let X exchange the two factors. The physical two-particle input space is the closed Bose subspace

$$\boxed{\mathcal{H}^{(2)}_{\mathrm{phys}} = \{\Phi \in \mathcal{H}_{\mathrm{pair}} : \mathsf{X}\Phi = \Phi\}.} \tag{29}$$

Paper 4 constructs one explicit nonzero unit vector in Eq. (29); the present analysis uses the ambient physical pair space as its starting datum.

2.2 Dense pair core and fixed-external-label source space

For the kernel calculation, let $\mathcal{D}_{\mathrm{pair}}$ denote the algebraic tensor product of finite separable Schwartz profiles used as the dense physical pair core. For each external one-particle label pair (a, b) there is a linear source map

$$\iota_{a,b} : \mathcal{D}_{\mathrm{pair}} \longrightarrow \mathcal{H}_{\mathrm{src}}, \tag{30}$$

where

$$\mathcal{H}_{\mathrm{src}} \simeq L^2(\mathbb{C} \times \mathbb{C}) \tag{31}$$

is the completed source Hilbert space at fixed external representation labels. In the joint source/output spectral representation it plays the role of a multiplicity space. Fixing (a, b) fixes the external helicities and frequencies; the vector in this source space remains variable:

$$\psi_{a,b} = \psi_{a,b}(z_1, z_2) \in L^2(\mathbb{C} \times \mathbb{C}). \tag{32}$$

The unitary $U^{\mathrm{src}}_{J,\nu}$ therefore acts on the source vector $\psi_{a,b}$ inside this fixed external-label sector.

The normalized physical Clebsch–Gordan coefficient is defined on this dense pair core. At fixed output label (J, ν) it produces an angular Hilbert-space vector,

$$C^{a,b,\mathrm{core}}_{J,\nu} : \mathcal{D}_{\mathrm{pair}} \longrightarrow \mathcal{H}^{\mathrm{ang}}_J. \tag{33}$$

The unconditional angular-transform statement used here is Eq. (33) on the dense physical pair core. Its completed direct-integral assembly, fixed-frequency bounded realization and range geometry are analytic questions carried forward to Paper 6.

2.3 Spectral representation of the source space

The source spectral coordinates arise from an explicit quotient chart on the two celestial source points. Away from the pair diagonal,

$$\boxed{(z_1, z_2) \longmapsto (z, \lambda) := \left(z_1, (z_1 - z_2)^{-1}\right),} \tag{34}$$

with inverse

$$(z, \lambda) \longmapsto \left(z, z - \lambda^{-1}\right). \tag{35}$$

Writing the nonzero quotient coordinate in logarithmic polar form

$$\lambda = e^{t+i\theta} \tag{36}$$

produces the source variables (z, t, θ). Fourier analysis in t and Fourier series in the periodic angle θ produce

$$(z, t, \theta) \longmapsto (z, \omega, k) \in \mathbb{C} \times (\mathbb{R} \times \mathbb{Z}). \tag{37}$$

The diagonal and polar seam form the standard null loci of these completed L^2 coordinate changes.

The completed transform is therefore

$$\boxed{\mathcal{F}_{\mathrm{src}} : \mathcal{H}_{\mathrm{src}} \xrightarrow{\simeq_u} L^2(X_{\mathrm{src}}, d\mu_{\mathrm{src}}),} \tag{38}$$

with

$$X_{\mathrm{src}} = \mathbb{C} \times (\mathbb{R} \times \mathbb{Z}), \qquad d\mu_{\mathrm{src}} = d^2 z\, d\omega\, d\#(k), \tag{39}$$

where $d\#$ is counting measure on $\mathbb{Z}$.

The source action in the shadow law becomes especially transparent in these coordinates. On the flat logarithmic cylinder its chord character is

$$\boxed{\chi_{J,\nu}(t,\theta) = e^{-2i\nu t}\, e^{-2iJ\theta}.} \tag{40}$$

With the Fourier convention used here,

$$e^{-2i\nu t} \quad \rightsquigarrow \quad \omega \longmapsto \omega + \frac{\nu}{\pi}, \qquad e^{-2iJ\theta} \quad \rightsquigarrow \quad k \longmapsto k + 2J. \tag{41}$$

Consequently, if $U^{\mathrm{src}}_{J,\nu}$ is the source-space action appearing in the kernel shadow law, then

$$\boxed{(\mathcal{F}_{\mathrm{src}} U^{\mathrm{src}}_{J,\nu} \mathcal{F}^{-1}_{\mathrm{src}} f)(z,\omega,k) = f\left(z, \omega + \frac{\nu}{\pi}, k + 2J\right).} \tag{42}$$

This is the geometric origin of the source shear used throughout the positive theorem.

2.4 Angular Clebsch–Gordan output

For $\Psi \in \mathcal{D}_{\mathrm{pair}}$, the normalized physical kernel row at (J,ν) lies in an angular Hilbert fiber,

$$C^{a,b,\mathrm{core}}_{J,\nu}[\Psi] \in \mathcal{H}^{\mathrm{ang}}_J. \tag{43}$$

The completed transform assembles such rows as almost-everywhere spectral data in a direct-integral output carrier. The retained angular fibers assemble into an angular-vector-valued Weyl-reduced coefficient space, denoted

$$\mathcal{H}^{\mathrm{red}}_{\mathrm{ang}}. \tag{44}$$

This is the codomain relevant to the actual Clebsch–Gordan application. It is useful to keep the source and output variables visibly separate:

$$\boxed{\underbrace{(z,\omega,k)}_{\text{source spectral data}} \quad \Big| \quad \underbrace{(J,\nu;w)}_{\text{output channel and angular variable}}, \qquad w \in S^2.} \tag{45}$$

The coordinate z retained by the Rühl source chart belongs to the input-pair geometry; w belongs to the output angular Hilbert fiber.

2.5 Joint source/output spectral space

The positive representation theorem uses the scalar joint object. Let

$$X_{\mathrm{joint}} = (\mathbb{Z} \times \mathbb{R}) \times X_{\mathrm{src}} \tag{46}$$

with

$$d\mu_{\mathrm{joint}} = d\#(J)\, d\nu\, d\mu_{\mathrm{src}}. \tag{47}$$

The scalar joint spectral Hilbert space is

$$\mathcal{H}_{\text{joint}} = L^2(X_{\text{joint}}, d\mu_{\text{joint}}). \tag{48}$$

For a fundamental output domain $\mathcal{F}$ defined below, the Weyl-reduced scalar space is

$$\boxed{\mathcal{H}_{\text{red}} = L^2(\mathcal{F} \times X_{\text{src}}, d\mu_{\text{joint}}|_{\mathcal{F}\times X_{\text{src}}}).} \tag{49}$$

The roles of the principal spaces are summarized as follows.

Object	Role	Mathematical form
$\mathcal{H}^{(2)}_{\text{phys}}$	physical state space	completed symmetric radiative pair space
$\mathcal{D}_{\text{pair}}$	analytic core	finite separable Schwartz pair tensors
$\mathcal{H}_{\text{src}}$	fixed-external-label source Hilbert space	completed L^2 space at fixed external labels
$L^2(X_{\text{src}}, \mu_{\text{src}})$	source spectrum	affine coordinate z, radial frequency ω, integer mode k
$\mathcal{H}^{\text{red}}_{\text{ang}}$	angular CG output	direct integral of retained angular Hilbert fibers
$\mathcal{H}_{\text{joint}}$	scalar joint representation space	signed output labels times source spectrum
$\mathcal{H}_{\text{red}}$	Weyl-reduced scalar space	one output fundamental domain times the full source spectrum

2.6 Hilbert and distributional domains

The singular star–triangle analysis also uses Schwartz distributions. The relevant hierarchy is

$$\boxed{\mathcal{S}(\mathbb{C}) \subset L^2(\mathbb{C}) \subset \mathcal{S}'(\mathbb{C}).} \tag{50}$$

The affine two-chord factor entering the singular kernel acts as a tempered distribution on Schwartz tests. Its inverse-distance punctures give logarithmic divergence of the squared norm on every puncture neighborhood, placing this factor in the distributional lane. The source vector $\psi \in \mathcal{H}_{\text{src}}$ remains an L^2 object. Pairing the distributional factor with controlled Schwartz tests, isolating the contact term and passing to the completed Hilbert-space matrix coefficient produces the completed transition.

The domain chain used later is therefore

$$\mathcal{D}_{\text{pair}} \xrightarrow{\iota_{a,b}} \mathcal{H}_{\text{src}} \xrightarrow{\mathcal{F}_{\text{src}}} L^2(X_{\text{src}}, \mu_{\text{src}}), \qquad \mathcal{S}(\mathbb{C}) \subset L^2(\mathbb{C}) \subset \mathcal{S}'(\mathbb{C}) \tag{51}$$

with the second chain governing the singular kernel factor and the first chain governing the physical source state.

3 Principal-series labels and the fundamental output sheet

The output labels are

$$(J,\nu) \in \mathbb{Z} \times \mathbb{R}. \tag{52}$$

The Weyl reflection is

$$w(J,\nu) = (-J,-\nu). \tag{53}$$

A convenient fundamental domain is

$$\boxed{\mathcal{F} = \{(J,\nu) : J > 0\} \cup \{(0,\nu) : \nu \geq 0\}.} \tag{54}$$

The unique output label fixed simultaneously by the fundamental-domain condition and its opposite is

$$(J,\nu) = (0,0). \tag{55}$$

Let

$$X^{+}_{\text{joint}} = \mathcal{F} \times X_{\text{src}}. \tag{56}$$

The output reflection exchanges X^{+}_{joint} with its complement up to the self-shadow null boundary in the direct-integral sense.

The principal-series Plancherel measure is invariant under

$$(J,\nu) \mapsto (-J,-\nu), \tag{57}$$

and the source translation introduced below is also measure preserving. These facts give equal measure weight to the retained and omitted sheets for vectors satisfying the exact fixed-sector law.

4 The normalized kernel and the source-aware shadow relation

Fix external one-particle labels a, b. For a core state $\Psi \in \mathcal{D}_{\text{pair}}$, write

$$C^{a,b,\text{core}}_{J,\nu}[\Psi] \in \mathcal{H}^{\text{ang}}_{J} \tag{58}$$

for the normalized physical Clebsch–Gordan coefficient row on the dense kernel core. The output shadow is a unitary operator

$$\mathcal{S}_{-\nu,J} : \mathcal{H}^{\text{ang}}_{-J} \xrightarrow{\simeq} \mathcal{H}^{\text{ang}}_{J}, \tag{59}$$

and the source side carries the unitary $U^{\text{src}}_{J,\nu}$ from Eq. (42). For the core state above write $\psi = \iota_{a,b}\Psi \in \mathcal{H}_{\text{src}}$. The all-label coefficient

$$\gamma_{a,b}(J,\nu) \in \mathbb{C} \tag{60}$$

satisfies

$$|\gamma_{a,b}(J,\nu)| = 1, \qquad \gamma_{a,b}(J,\nu)\gamma_{a,b}(-J,-\nu) = 1. \tag{61}$$

The source-aware shadow relation follows from a branch-safe identity of the normalized kernel. Define the unit-modulus Naimark character

$$\chi_{J,\nu}(u) := \mathcal{N}(u; 2i\nu, 2J), \qquad u \in \mathbb{C} \setminus \{0\}, \tag{62}$$

where $\mathcal{N}$ is the normalized radial/helicity character used in the Clebsch–Gordan kernel. It obeys

$$|\chi_{J,\nu}(u)| = 1, \qquad \chi_{-J,-\nu}(u)\chi_{J,\nu}(u) = 1. \tag{63}$$

On the source Hilbert space,

$$\boxed{(U^{\mathrm{src}}_{J,\nu}\psi)(z_1, z_2) = \chi_{J,\nu}(z_1 - z_2)\psi(z_1, z_2)} \quad \text{a.e.} \tag{64}$$

Thus $U^{\mathrm{src}}_{J,\nu}$ is a unitary multiplication operator, and the opposite-label character gives its inverse.

Let $K^{a,b}_{J,\nu}(z_1, z_2; w)$ denote the normalized kernel on the regular locus $z_1 \neq z_2$, $z_2 \neq w$, $w \neq z_1$. The three Naimark factors in the kernel satisfy the exact transition identity

$$\boxed{K^{a,b}_{J,\nu}(z_1, z_2; w)\,\chi_{J,\nu}(z_1 - z_2) = K^{a,b}_{-J,-\nu}(z_1, z_2; w)\,\chi_{J,\nu}(z_2 - w)\,\chi_{J,\nu}(w - z_1).} \tag{65}$$

This identity is obtained by adding $(2i\nu, 2J)$ to the input-input exponent and subtracting the same pair from the two output-chord exponents. Because the helicity exponent is integral, the multiplicative character law is branch-cocycle free.

The two output-chord characters in Eq. (65) define the completed output transition operator. Its star–triangle evaluation is

$$\boxed{\mathcal{R}^{a,b}_{J,\nu} = \gamma_{a,b}(J, \nu)\,\mathcal{S}_{-\nu,J},} \tag{66}$$

where $\mathcal{R}^{a,b}_{J,\nu}$ denotes the completed residual two-chord operator. The phase can be written

$$\gamma_{a,b}(J, \nu) = (-1)^J \pi^{-2i\nu}\, G^{a,b}_0(J, \nu) G^{a,b}_1(J, \nu). \tag{67}$$

To expose the convention completely, let $j_a, j_b \in \{\pm 2\}$ be the external celestial spins and let ν_a, ν_b be their principal-series frequencies from Eq. (23). Define

$$A^{a,b}_0(J) = \frac{1 - J + j_b - j_a}{2}, \qquad B^{a,b}_0(\nu) = \frac{\nu + \nu_b - \nu_a}{2}, \tag{68}$$

$$A^{a,b}_1(J) = \frac{1 - J + j_a - j_b}{2}, \qquad B^{a,b}_1(\nu) = \frac{\nu + \nu_a - \nu_b}{2}. \tag{69}$$

On the regular Gamma locus the two phase ratios are

$$\boxed{G^{a,b}_r(J, \nu) = \frac{\Gamma\Big(A^{a,b}_r(J) + iB^{a,b}_r(\nu)\Big)}{\Gamma\Big(A^{a,b}_r(J) - iB^{a,b}_r(\nu)\Big)}, \qquad r \in \{0, 1\}.} \tag{70}$$

At the isolated pole-collision parameters, the completed construction assigns its fixed resolved phase value. Denoting that continuation by

$$\mathfrak{G}(A, B) = \begin{cases} -1, & \text{at the resolved collision,} \\ \Gamma(A + iB)/\Gamma(A - iB), & \text{otherwise,} \end{cases} \tag{71}$$

we have simply

$$G^{a,b}_0(J, \nu) = \mathfrak{G}(A^{a,b}_0(J), B^{a,b}_0(\nu)), \qquad G^{a,b}_1(J, \nu) = \mathfrak{G}(A^{a,b}_1(J), B^{a,b}_1(\nu)). \tag{72}$$

For real A_r, B_r the numerator and denominator in Eq. (70) are conjugate values, so each regular ratio has unit modulus; the resolved collision value does as well. Consequently

$$|G_r^{a,b}(J,\nu)| = 1, \qquad |\gamma_{a,b}(J,\nu)| = 1, \tag{73}$$

and the completed opposite-label identity gives

$$\gamma_{a,b}(J,\nu)\gamma_{a,b}(-J,-\nu) = 1. \tag{74}$$

Equations (65) and (66) determine the source unitary, output shadow and channel phase entering the bounded source-aware operator-family class. For every row family $\mathfrak{C} = \{\mathcal{C}_{J,\nu}\}$ in this class,

$$\boxed{\mathcal{C}_{J,\nu}[\psi] = \gamma_{a,b}(J,\nu)\, \mathcal{S}_{-\nu,J}\mathcal{C}_{-J,-\nu}[U^{\mathrm{src}}_{J,\nu}\psi].} \tag{75}$$

The reconstruction law includes the scalar row and every nonzero integral-spin row.

Unitarity of the shadow operator and $|\gamma_{a,b}(J,\nu)| = 1$ give

$$\boxed{\|\mathcal{C}_{J,\nu}[\psi]\| = \|\mathcal{C}_{-J,-\nu}[U^{\mathrm{src}}_{J,\nu}\psi]\|.} \tag{76}$$

Hence the Weyl-paired two-row energy is

$$\boxed{E_{\mathrm{Weyl}}(J,\nu;\psi) := \|\mathcal{C}_{J,\nu}[\psi]\|^2 + \|\mathcal{C}_{-J,-\nu}[U^{\mathrm{src}}_{J,\nu}\psi]\|^2 = 2\|\mathcal{C}_{J,\nu}[\psi]\|^2.} \tag{77}$$

The literal core row $C^{a,b,\mathrm{core}}_{J,\nu}[\Psi]$ and the bounded row $\mathcal{C}_{J,\nu}$ therefore occupy distinct analytic levels. The kernel identities supply the representation data used by $\mathfrak{C}$; realization of an everywhere-defined bounded row by the physical direct-integral coefficient is an additional analytic problem carried forward to Paper 6.

4.1 Source-aware operator families

The completed source-aware operator-family class is

$$\mathfrak{C} = \{\mathcal{C}_{J,\nu}\}_{(J,\nu)\in\mathbb{Z}\times\mathbb{R}}, \qquad \mathcal{C}_{J,\nu} : \mathcal{H}_{\mathrm{src}} \longrightarrow \mathcal{H}^{\mathrm{flat}}. \tag{78}$$

Every family in this class satisfies Eq. (75). Equality on the fundamental output sheet determines every omitted signed row, and every retained-row family extends through the explicit source unitary, output shadow and channel phase. Fundamental-row restriction is therefore a bijection on this bounded operator-family class.

The symbol $\mathcal{C}_{J,\nu}$ is reserved for these everywhere-defined bounded rows. The symbol $C^{a,b,\mathrm{core}}_{J,\nu}$ denotes the physical dense-core Clebsch–Gordan coefficient. Their analytic realization relation belongs to the completed angular-transform problem treated in Paper 6.

5 Joint spectral shear

The spectral representation in Eq. (38) turns the source action into an explicit coordinate translation. Define

$$\boxed{\Theta_{J,\nu}(z,\omega,k)=\left(z,\omega+\frac{\nu}{\pi},k+2J\right).} \tag{79}$$

Equation (42) can then be written

$$\mathcal{F}_{\mathrm{src}}U^{\mathrm{src}}_{J,\nu}\mathcal{F}_{\mathrm{src}}^{-1}=\Theta^{*}_{J,\nu} \tag{80}$$

in the almost-everywhere L^2 sense, where $\Theta^{*}_{J,\nu}f=f\circ\Theta_{J,\nu}$.

The inverse relation is immediate from the coordinates:

$$\begin{aligned}\Theta_{-J,-\nu}(\Theta_{J,\nu}(z,\omega,k))&=\Theta_{-J,-\nu}\left(z,\omega+\frac{\nu}{\pi},k+2J\right) && (81)\\ &=\left(z,\omega+\frac{\nu}{\pi}-\frac{\nu}{\pi},k+2J-2J\right) && (82)\\ &=(z,\omega,k). && (83)\end{aligned}$$

Hence

$$\boxed{\Theta_{-J,-\nu}\circ\Theta_{J,\nu}=\mathbf{1}.} \tag{84}$$

The source measure is also preserved. The affine coordinate is unchanged, Lebesgue measure is translation invariant in ω, and counting measure is translation invariant in k:

$$(\tau_{\nu/\pi})_{*}d\omega=d\omega,\qquad (s_{2J})_{*}d\#=d\#. \tag{85}$$

Therefore

$$\boxed{(\Theta_{J,\nu})_{*}\mu_{\mathrm{src}}=\mu_{\mathrm{src}}.} \tag{86}$$

The joint Weyl shear on

$$X_{\mathrm{joint}}=(\mathbb{Z}\times\mathbb{R})\times\mathbb{C}\times(\mathbb{R}\times\mathbb{Z}) \tag{87}$$

is now completely explicit:

$$\boxed{\mathscr{W}((J,\nu),(z,\omega,k))=\left((-J,-\nu),\left(z,\omega+\frac{\nu}{\pi},k+2J\right)\right).} \tag{88}$$

Applying Eq. (88) twice uses the reflected parameters $(-J,-\nu)$ on the second step, so Eq. (84) gives

$$\boxed{\mathscr{W}^2=\mathbf{1}.} \tag{89}$$

Reflection preserves counting measure in J and Lebesgue measure in ν; Eq. (86) preserves the source factor. Thus

$$\boxed{\mathscr{W}_{*}\mu_{\mathrm{joint}}=\mu_{\mathrm{joint}}.} \tag{90}$$

For fixed external labels, define the phase-bearing pullback

$$(\mathsf{W}_{a,b}F)(x)=\gamma_{a,b}(J,\nu)F(\mathscr{W}x),\qquad x=((J,\nu),(z,\omega,k)). \tag{91}$$

The unit-modulus phase and Eq. (90) make $\mathsf{W}_{a,b}$ unitary. The opposite-phase identity in Eq. (61) and Eq. (89) give

$$\mathsf{W}_{a,b}^2 = \mathbf{1}. \tag{92}$$

The Weyl-fixed subspace is

$$\boxed{\mathcal{H}_{\text{joint}}^{\text{fix}}(a,b) = \{F \in \mathcal{H}_{\text{joint}} : \mathsf{W}_{a,b}F = F\}.} \tag{93}$$

Equivalently,

$$F(J,\nu,z,\omega,k) = \gamma_{a,b}(J,\nu)F\Big(-J,-\nu,z,\omega+\frac{\nu}{\pi},k+2J\Big) \quad \text{a.e.} \tag{94}$$

The nontrivial positive content is therefore the exact identification of the physical source action with the translation in Eq. (79); the subsequent two-sheet Hilbert-space reduction uses this explicit measure-preserving involution.

6 Weyl reduction and normalized restriction

Let

$$R : \mathcal{H}_{\text{joint}} \longrightarrow \mathcal{H}_{\text{red}} \tag{95}$$

be restriction to X_{joint}^{+}, and let

$$E : \mathcal{H}_{\text{red}} \longrightarrow \mathcal{H}_{\text{joint}} \tag{96}$$

be zero extension.

For $F, G \in \mathcal{H}_{\text{joint}}^{\text{fix}}(a,b)$, invariance under $\mathsf{W}_{a,b}$ and measure preservation give equal contributions from the retained and omitted sheets:

$$\langle F, G\rangle_{\text{joint}} = 2\langle RF, RG\rangle_{\text{mf}}. \tag{97}$$

The normalization is therefore forced:

$$\boxed{\mathcal{A}_{a,b}F = \sqrt{2}\,RF.} \tag{98}$$

Then

$$\langle \mathcal{A}_{a,b}F, \mathcal{A}_{a,b}G\rangle_{\text{mf}} = \langle F, G\rangle_{\text{joint}}. \tag{99}$$

The $\sqrt{2}$ normalization in Eq. (98) is the Hilbert-space form of the transformed-source pairing in Eq. (77). For one unchanged source vector, the signed-row energy is the distinct quantity

$$E_{\text{fixed}}(J,\nu;\psi) := \big\|\mathcal{C}_{J,\nu}[\psi]\big\|^2 + \big\|\mathcal{C}_{-J,-\nu}[\psi]\big\|^2. \tag{100}$$

On source-fixed vectors,

$$U_{J,\nu}^{\text{src}}\psi = \psi \quad \Longrightarrow \quad E_{\text{fixed}}(J,\nu;\psi) = E_{\text{Weyl}}(J,\nu;\psi). \tag{101}$$

The Weyl-reduction theorem uses E_{Weyl} and the joint fixed-subspace law.

The corresponding synthesis map is

$$\boxed{\mathcal{S}_{a,b} f = \frac{1}{\sqrt{2}} \left(Ef + \mathsf{W}_{a,b} Ef \right).} \tag{102}$$

By construction,

$$\mathsf{W}_{a,b} \mathcal{S}_{a,b} f = \mathcal{S}_{a,b} f, \tag{103}$$

so synthesis lands in the Weyl-fixed subspace.

The two restriction identities are

$$R \, \mathsf{W}_{a,b} E = 0, \tag{104}$$

and

$$RE = \mathbf{1}. \tag{105}$$

Consequently

$$\boxed{\mathcal{A}_{a,b} \mathcal{S}_{a,b} = \mathbf{1}_{\mathcal{H}_{\mathrm{red}}},} \tag{106}$$

and the fixed-sector law gives

$$\boxed{\mathcal{S}_{a,b} \mathcal{A}_{a,b} = \mathbf{1}_{\mathcal{H}^{\mathrm{fix}}_{\mathrm{joint}}(a,b)}.} \tag{107}$$

Thus

$$\boxed{\mathcal{A}_{a,b} : \mathcal{H}^{\mathrm{fix}}_{\mathrm{joint}}(a,b) \xrightarrow{\simeq_u} \mathcal{H}_{\mathrm{red}}.} \tag{108}$$

What Eq. (108) means. The domain and codomain are scalar L^2 representation spaces on the joint source/output spectrum. Equation (108) is therefore a Plancherel theorem for this Weyl-fixed joint spectral representation. A Plancherel theorem for the angular Clebsch–Gordan application would require an isometric completed map

$$\mathcal{H}_{\mathrm{src}} \longrightarrow \mathcal{H}^{\mathrm{red}}_{\mathrm{ang}} \tag{109}$$

with its own dense-core agreement, norm identity and range theorem. The unconditional theorem established here closes at the scalar Weyl-fixed joint spectral representation.

7 External-label measurability and global phase assembly

The external labels include the two incoming helicities and source frequencies, the integral output spin and the output principal-series frequency. Schematically,

$$\lambda_{\mathrm{ext}} = (\lambda_1, \lambda_2, J; \omega_1, \omega_2, \nu). \tag{110}$$

The exact all-label coefficient

$$\gamma(\lambda_{\mathrm{ext}}) \tag{111}$$

is jointly continuous in the three real frequency variables on every discrete helicity/spin sector and is therefore measurable on the full joint external/output spectral parameter space.

Multiplication by this phase defines a global unitary on the corresponding external L^2 space:

$$(\mathsf{M}_\gamma F)(\lambda_{\text{ext}}) = \gamma(\lambda_{\text{ext}})F(\lambda_{\text{ext}}), \tag{112}$$

with

$$\|\mathsf{M}_\gamma F\| = \|F\| \tag{113}$$

and explicit inverse

$$\mathsf{M}_\gamma^{-1} = \mathsf{M}_{\gamma^{-1}}. \tag{114}$$

Together with the measurable joint shear, this establishes the global measurable assembly of the source-aware phase-bearing representation.

8 Weyl-invariant joint spectral subspace

The representation-theoretic Hilbert-space theorem is

$$\boxed{\mathcal{H}_{\text{joint}}^{\text{fix}}(a,b) \simeq_u \mathcal{H}_{\text{red}}} \tag{115}$$

for every fixed external-label pair (a,b), with the exact source-twisted Weyl law, measurable phase assembly and explicit analysis/synthesis formulas.

Interpretive boundary. Equation (115) is the Plancherel theorem for the scalar Weyl-fixed joint spectral representation constructed in this paper. The angular Clebsch–Gordan application occupies the completed map

$$\mathcal{H}_{\text{src}} \longrightarrow \mathcal{H}_{\text{ang}}^{\text{red}}, \tag{116}$$

whose full Plancherel classification additionally requires dense-core agreement, an exact norm identity and an identified range. The theorem established here closes exactly on the scalar Weyl-fixed joint spectral representation.

Its mathematical role is:

$$\boxed{\begin{array}{c}\mathcal{H}_{\text{red}} = L^2(\mathcal{F} \times X_{\text{src}}),\\ \text{with the full source Hilbert space retained.}\end{array}} \tag{117}$$

On the dense kernel core, the physical angular Clebsch–Gordan output is assembled as an almost-everywhere direct-integral class,

$$\Psi \longmapsto \Big[(J,\nu) \longmapsto C_{J,\nu}^{a,b,\text{core}}[\Psi]\Big]_{L^2}, \tag{118}$$

with

$$C_{J,\nu}^{a,b,\text{core}}[\Psi] \in \mathcal{H}_J^{\text{ang}} \quad \text{for the selected measurable representative and almost every spectral label.} \tag{119}$$

The brackets in Eq. (118) denote the completed L^2 equivalence class. The scalar coordinates of $\mathcal{H}_{\text{red}}$ contain the retained output label and source spectrum, while the angular Hilbert variable belongs to the physical output carrier.

Accordingly, the paper records the type-level separation

$$\boxed{\mathcal{H}_{\mathrm{red}} \quad \text{and} \quad \mathcal{H}_{\mathrm{ang}}^{\mathrm{red}} \quad \text{carry their respective exact roles.}} \tag{120}$$

This classification fixes the interpretation of the positive theorem and preserves the established analysis, synthesis and Parseval identities.

9 Hilbert-space distinctions and normalization checks

The normalization and quotient tests below characterize the exact relation between the physical weighted channels, their flat-frequency representatives and the Weyl-fixed reduction.

9.1 Exact Plancherel flattening

Equation (25) absorbs the full fixed-spin density into the coefficient itself. Writing

$$w_J(\nu) = \sqrt{8(J^2 + \nu^2)}, \tag{121}$$

the energy identity is

$$\boxed{\int_{\mathbb{R}} |f(\nu)|^2\, 8(J^2 + \nu^2)\, d\nu = \int_{\mathbb{R}} |w_J(\nu) f(\nu)|^2\, d\nu.} \tag{122}$$

The inverse-Plancherel half-density appearing in the normalized Clebsch–Gordan coefficient is the inverse of this square-root weight. Together with the angular-period factor 2π, this fixes the normalization used in the physical coefficient.

9.2 Raw and Plancherel-normalized coefficients

The normalized coefficient contains the angular-period factor and the inverse Plancherel half-density. At the retained label $(J, \nu) = (1, 0)$ the resulting scalar is nontrivial. For every nonzero raw row,

$$C_{1,0}^{\mathrm{raw}} \neq 0 \quad \Longrightarrow \quad C_{1,0}^{\mathrm{norm}} \neq C_{1,0}^{\mathrm{raw}}. \tag{123}$$

Thus the normalized Clebsch–Gordan coefficient carries a genuinely modified representative on every nonzero row covered by this calculation.

9.3 Restriction of unrestricted signed data

Consider the full signed output space and let D discard one member of each reflected pair. The completed comparison map supplies an explicit nonzero source $\Psi_\star$ with

$$\boxed{\Psi_\star \neq 0, \qquad D\Psi_\star = 0, \qquad \ker D \neq \{0\}.} \tag{124}$$

Equation (124) shows that unrestricted signed data retain a nontrivial kernel. Invertibility is a property of restriction on the Weyl-fixed subspace.

The two-sheet rescaling obeys $\ker(\sqrt{2}D) = \ker D$, preserving the same nontrivial kernel.

9.4 Distributional affine two-chord factor

The affine two-chord distributional factor acts weakly on Schwartz tests. Its inverse-distance punctures give logarithmically divergent squared norm on every puncture neighborhood. Consequently

$$\text{affine two-chord distribution} \notin L^2_{\mathrm{Pl}} \tag{125}$$

at that stage. The subsequent convolution/Fourier analysis consequently proceeds in the distributional setting.

9.5 Same-source phase-only comparison

The exact source-twisted operator-family law specializes on source-fixed vectors:

$$\boxed{U^{\mathrm{src}}_{J,\nu}\psi = \psi \quad \Longrightarrow \quad \mathcal{C}_{J,\nu}[\psi] = \gamma_{J,\nu}\, \mathcal{S}_{-\nu,J}\mathcal{C}_{-J,-\nu}[\psi].} \tag{126}$$

A complete-channel same-source contract promotes the right-hand side of Eq. (126) to every source vector. At the spin-one zero-frequency row, the source action is nontrivial, and the simultaneous complete-channel phase-only and source-twisted contracts obey

$$\boxed{\neg C_{\mathrm{phase}}}. \tag{127}$$

The source-fixed specialization and the complete-channel no-go therefore occupy distinct scopes.

10 Fixed-source isometry and the unit-Gram condition

Let

$$\mathcal{T}^{a,b}_{J,\nu} \tag{128}$$

denote the contact-normalized Clebsch–Gordan transition coefficient at fixed external labels. After source and observer amplitudes have been factored out, the corresponding Gram form is multiplied by

$$\boxed{G_{\mathrm{out}}(J,\nu) = |\mathcal{T}^{a,b}_{J,\nu}|^2\, G_{\mathrm{in}}(J,\nu).} \tag{129}$$

The unit-Gram condition follows directly from isometry. Let $V : \mathcal{H}_s \to \mathcal{H}_t$ be a linear isometry and suppose unit vectors e_s, e_t satisfy

$$V e_s = \mathcal{T}\, e_t, \qquad \|e_s\| = \|e_t\| = 1. \tag{130}$$

Then

$$1 = \|e_s\|^2 = \|V e_s\|^2 = \|\mathcal{T} e_t\|^2 = |\mathcal{T}|^2. \tag{131}$$

Hence every fiberwise isometric realization of the normalized Clebsch–Gordan transition requires

$$\boxed{|\mathcal{T}^{a,b}_{J,\nu}|^2 = 1.} \tag{132}$$

This necessity statement is independent of the historical multi-condition formulation discussed in Section 16.

11 Distributional completion and contact normalization

The singular variable in this section is the output-shadow angular coordinate

$$w \in \mathbb{C}_{\mathrm{sh}}, \qquad \mathcal{S}(\mathbb{C}_{\mathrm{sh}}) \subset L^2(\mathbb{C}_{\mathrm{sh}}) \subset \mathcal{S}'(\mathbb{C}_{\mathrm{sh}}). \tag{133}$$

All finite-part extensions, Dirac contact terms and homogeneity conditions below are distributions in this output-shadow plane. The normalized kernel contains singular homogeneous factors. Its completion is organized by punctured finite-part distributions, contact terms and compatible cofinal cutoff families. The contact factor is the canonical nonzero scalar carried by the coincidence channel of the completed distributional kernel. Dividing the completed pair-core scalar by this factor removes the universal contact normalization and leaves the dimensionless transition $\mathcal{T}^{a,b}_{J,\nu}$ whose Gram multiplier appears in Eq. (129):

$$\boxed{\mathcal{T}^{a,b}_{J,\nu} = \frac{\mathcal{M}^{a,b}_{J,\nu,\mathrm{comp}}}{A^{a,b}_{\mathrm{contact}}(J,\nu)}.} \tag{134}$$

The relevant analytic stages have the following structure.

First, the normalized finite-part distribution is paired with Schwartz test functions, with the contact contribution displayed as a separate term. The finite regularization decomposes as

$$\mathcal{I}_{\mathrm{raw}} = \mathcal{I}_{\mathrm{fp}} + \mathcal{D}_{\mathrm{contact}}. \tag{135}$$

The contact defect admits a cofinal exhaustion with the explicit reciprocal bound

$$|\mathcal{D}^{(n)}_{\mathrm{contact}}| \leq \frac{1}{n+1}, \tag{136}$$

and therefore vanishes along the compatible cofinal cutoff family.

Second, the finite operator cutoff and outer Schwartz exhaustion can be synchronized. The resulting one-index diagonal satisfies a stagewise estimate of the schematic form

$$|\mathcal{M}_n - \mathcal{M}_\infty| \leq \frac{2}{n+1}, \tag{137}$$

for the physical matrix coefficient. Two source coefficients share one synchronized schedule, enabling polarization directly on a common finite family.

Third, the completed pair-core scalar factors into source amplitudes, observer factors and one universal label-dependent scalar. Division by the nonzero contact factor gives the normalized scalar in Eq. (128).

These limits identify the spin-one calculation with the completed normalized transition.

12 Spin-one zero-frequency evaluation

We now specialize to equal positive external helicities at zero external frequency and to the composite label

$$(J,\nu) = (1,0). \tag{138}$$

Here J is the spin of the *composite output representation*; the external graviton spins remain $j_a = j_b = +2$. The choice $J = 1$ is the first nonzero composite integral-spin row, and $\nu = 0$ gives the simplest real-frequency representative of that row. The universal all-label Gram condition quantifies over every external pair and every output spin, so a positive-measure interval attached to this single external pair and spin decides the universal condition.

12.1 Unweighted transition skeleton

The rational transition skeleton separates into three radial regions with thresholds

$$r = 1, \qquad r = 2. \tag{139}$$

The exact angular means on those regions are

$$\boxed{\frac{7}{8}, \qquad -\frac{1}{8}, \qquad 0.} \tag{140}$$

The polar moment is

$$\boxed{\frac{\pi}{2}.} \tag{141}$$

After the normalized shadow and contact factors are included, the unweighted completed coefficient is the exact phase

$$\boxed{-1}. \tag{142}$$

12.2 Normalized spherical outer state and independent normalization ledger

The completed pair-core problem contains the normalized round-sphere outer half-density. The round-sphere constant wave has area 4π, so its normalized amplitude is $(4\pi)^{-1/2}$. Transport through the sphere-to-plane density unitary gives a flat representative of norm one. The corresponding unnormalized outer factor has norm $\sqrt{4\pi}$.

A dependency-independent reconstruction beginning from the physical two-particle Clebsch–Gordan kernel derives the complete spin-one normalization ledger

$$\boxed{N_{\mathrm{CG}} = \frac{1}{8\pi^2}, \qquad \rho_{\mathrm{Pl}}(1,0) = 8, \qquad P_{\mathrm{ang}} = 2\pi, \qquad S_{1,0} = -\frac{1}{\pi}, \qquad A_{\mathrm{contact}}(0) = \frac{N_{\mathrm{CG}}}{2}.} \tag{143}$$

The weighted shadow–outer–transition product therefore carries the contact-normalized prefactor

$$\boxed{\frac{-2N_{\mathrm{CG}}/\pi}{N_{\mathrm{CG}}/2} = -\frac{4}{\pi}.} \tag{144}$$

Write the radial-square change of variables as

$$r(x)^2 = \frac{x(2-x)}{1-x}, \qquad 0 < x < 1. \tag{145}$$

The moving poles cross $r = 1$ and $r = 2$ at

$$\boxed{x_1 = \frac{3-\sqrt{5}}{2}, \qquad x_2 = 3-\sqrt{5}, \qquad 0 < x_1 < x_2 < 1.} \tag{146}$$

The spherical half-density reduces the residue to three elementary radial pieces,

$$M = \int_0^{x_1} f_{\mathrm{I}}(x)\,dx + \int_{x_1}^{x_2} f_{\mathrm{M}}(x)\,dx + \int_{x_2}^{1} f_{\mathrm{O}}(x)\,dx. \tag{147}$$

Appendix D displays $f_{\mathrm{I}}, f_{\mathrm{M}}, f_{\mathrm{O}}$ and elementary antiderivatives. Direct boundary evaluation gives

$$\boxed{M = -\frac{1}{8} + \frac{1}{2}\log 2.} \tag{148}$$

The weighted rational kernel has a second-order contact puncture. Its global completion is the contour-selected finite-part plane functional

$$I_{\mathrm{sph}}^{\mathrm{FP}} = 2\pi M. \tag{149}$$

The independently reconstructed normalization in Eq. (144) then gives

$$\mathcal{T}_{1,0} = \left(-\frac{4}{\pi}\right) I_{\mathrm{sph}}^{\mathrm{FP}} = \left(-\frac{4}{\pi}\right)(2\pi M) = -8M. \tag{150}$$

Hence

$$\boxed{\mathcal{T}_{1,0} = 1 - 4\log 2.} \tag{151}$$

The contact-cancelled weighted-minus-unweighted relative integrand is ordinarily integrable. The independent reconstruction identifies it almost everywhere with the unevaluated relative-defect integrand of the completed normalized transition. Integration then gives the structural equality

$$\boxed{\Delta_{\mathrm{rel}}^{\mathrm{ind}} = \Delta_{\mathrm{rel}}^{\mathrm{actual}}.} \tag{152}$$

The direct unweighted contour calculation independently recovers the residue table $7/8, -1/8, 0$ and the phase -1. The two derivations therefore meet at the same completed transition before the final closed-form evaluation.

The relative correction to the unweighted phase is

$$\boxed{\Delta_{\mathrm{rel}} = 2 - 4\log 2.} \tag{153}$$

Since $\log 2 > 1/2$,

$$4\log 2 > 2, \tag{154}$$

so

$$1 - 4\log 2 < -1 \tag{155}$$

and therefore

$$\boxed{|1 - 4\log 2| > 1.} \tag{156}$$

Thus

$$\boxed{|\mathcal{T}_{1,0}|^2 \neq 1.} \tag{157}$$

The independent original-kernel route and the completed pair-core route meet before the final scalar is inserted. The dependency-independent calculation recomputes the Clebsch–Gordan exponents, contour residues, moving-pole geometry and kernel-to-density identities. Its relative integrand is then identified almost everywhere with the unevaluated completed relative-defect integrand. Together with the structurally bound unweighted finite part, this gives

$$\boxed{\mathcal{T}_{1,0}^{\mathrm{ind}} = \mathcal{T}_{1,0}^{\mathrm{comp}} = 1 - 4\log 2.} \tag{158}$$

Appendix D.1 records the normalization and finite-part anatomy of this validation.

13 Positive-measure obstruction to fixed-source isometry

The spin-one value in Eq. (151) is initially evaluated at zero incoming Mellin frequencies and zero output frequency. The direct-integral statement requires all three continuous spectral coordinates.

For positive external helicity define

$$a_+(\nu_1) := \left(+2, \frac{\nu_1}{2\pi}\right), \qquad b_+(\nu_2) := \left(+2, \frac{\nu_2}{2\pi}\right), \tag{159}$$

so their one-particle Mellin frequencies are exactly ν_1 and ν_2. Define the spin-one transition

$$\boxed{\mathcal{T}_3(\nu_1, \nu_2, \nu) := \mathcal{T}_{1,\nu}^{a_+(\nu_1), b_+(\nu_2)}.} \tag{160}$$

At the origin,

$$\mathcal{T}_3(0, 0, 0) = 1 - 4\log 2. \tag{161}$$

The continuity proof keeps the normalized physical kernel throughout. The incoming conformal weights and radial exponents depend continuously on (ν_1, ν_2, ν). On a compact cube around the origin, all relevant Gamma denominators remain in a pole-free region and the contact amplitude remains nonzero. A compact C^1 estimate supplies one Lipschitz constant for the normalized contact subtraction on the cube. Together with the explicit puncture and exterior envelopes of Appendix E, these estimates provide common $L^1(\mathbb{C})$ majorants for the finite-part and relative-correction integrands. Dominated convergence gives

$$\boxed{\mathrm{ContinuousAt}\, \mathcal{T}_3(0, 0, 0).} \tag{162}$$

Set

$$G_3(\nu_1, \nu_2, \nu) := |\mathcal{T}_3(\nu_1, \nu_2, \nu)|^2. \tag{163}$$

Since

$$G_3(0, 0, 0) = |1 - 4\log 2|^2 > 1, \tag{164}$$

continuity yields $\epsilon > 0$ such that

$$\boxed{G_3(\nu_1, \nu_2, \nu) > 1 \qquad \text{for every } (\nu_1, \nu_2, \nu) \in B_\epsilon(0) \subset \mathbb{R}^3.} \tag{165}$$

The ball $B_\epsilon(0)$ has positive three-dimensional Lebesgue measure. The fixed discrete row

$$(\lambda_1, \lambda_2, J) = (+2, +2, 1) \tag{166}$$

has counting mass one. Therefore the defect set has positive measure with respect to the joint external/output spectral measure.

Consequently the almost-everywhere condition

$$|\mathcal{T}|^2 = 1 \quad \text{a.e. in } (\nu_1, \nu_2, \nu) \tag{167}$$

fails on the joint external/output spectral parameter space. Combining Eq. (165) with the isometry implication in Eq. (131) gives the principal negative theorem:

$$\boxed{\begin{array}{c}\text{no almost-everywhere fixed-source isometric realization}\\ \text{agrees with the normalized physical kernel.}\end{array}} \tag{168}$$

14 Completion rigidity and finite contact freedom

The distributional calculation separates two kinds of freedom: the choice of normalized cutoff used to represent the canonical shadow extension, and the addition of a finite point-supported term to an extension defined from regular-locus data.

For any two admissible normalized shadow cutoffs c_1, c_2, the finite-part integral and contact coefficient transform oppositely. The completed value is therefore cutoff independent:

$$\boxed{\mathcal{M}^{c_1}_{\text{comp}} = \mathcal{M}^{c_2}_{\text{comp}}.} \tag{169}$$

At the audited spin-one zero-frequency row, every tempered extension agreeing with the punctured kernel on tests vanishing at the contact has the unique form

$$\boxed{D_c = D_{\text{can}} + c\,\delta_0, \qquad c \in \mathbb{C}.} \tag{170}$$

The extension ambiguity is exactly one-dimensional, generated by δ_0. The normalized transition responds as

$$\boxed{\mathcal{T}_c = \mathcal{T}_{\text{can}} + c.} \tag{171}$$

Thus regular-locus agreement alone leaves modulus-changing freedom. Unit-modulus representatives satisfy

$$c = e^{i\alpha} - (1 - 4\log 2), \qquad \alpha \in \mathbb{R}, \tag{172}$$

and the explicit choice

$$c = 4\log 2 \tag{173}$$

gives $\mathcal{T}_c = 1$.

Compatibility with the homogeneous shadow-extension law imposes the following covariance condition. If $\mathcal{D}_\lambda$ denotes dilation of the Schwartz test by $\lambda \in \mathbb{C}^\times$, then admissibility requires

$$\boxed{D_c(\mathcal{D}_\lambda \varphi) = m_{0,1}(\lambda)\, D_c(\varphi) \qquad (\lambda \in \mathbb{C}^\times),} \tag{174}$$

where $m_{0,1}$ is the spin-one shadow dilation character. The exact extension theorem gives

$$\boxed{\text{Eq. (174) holds} \iff c = 0.} \tag{175}$$

The canonical contact-probe and Fourier normalizations select the same zero coordinate. Hence every extension in the homogeneous/Fourier-normalized class has

$$\boxed{\mathcal{T}_{\rm phys} = \mathcal{T}_{\rm can}, \qquad |\mathcal{T}_{\rm phys}|^2 = |\mathcal{T}_{\rm can}|^2.} \tag{176}$$

Regular-locus extension theory therefore has one complex contact coordinate, and the homogeneous/Fourier-normalized completion has a unique representative.

15 Source representation and source-frame trivialization

The unitary source action in Eq. (75) is a strongly continuous representation of

$$\Gamma := \mathbb{Z} \times \mathbb{R}. \tag{177}$$

For $g = (J, \nu)$ write $U_g := U^{\rm src}_{J,\nu}$. Then

$$\boxed{U_0 = I, \qquad U_g U_h = U_{g+h}, \qquad U_{-g} = U_g^{-1},} \tag{178}$$

and $\nu \mapsto U_{(J,\nu)}\psi$ is continuous for every fixed J and $\psi \in \mathcal{H}_{\rm src}$.

On a nonzero source chord q,

$$\boxed{\chi_{J,\nu}(q) = \|q\|^{2i\nu} \left(\frac{q}{\|q\|}\right)^{2J}.} \tag{179}$$

This is the source-space character whose log-polar Fourier transform yields Eq. (79).

A *source-frame gauge* is a label-dependent unitary change of trivialization

$$V_g \in \mathcal{U}(\mathcal{H}_{\rm src}).$$

This terminology concerns the source-space bundle over spectral labels and is unrelated to spacetime gauge symmetry. The transformed source action is

$$\boxed{U_g^V = V_{-g}\, U_g\, V_g^{-1}.} \tag{180}$$

Three classes are useful. Scalar row rephasings alter only the scalar cocycle and preserve

$$|e^{i\alpha(g)}\mathcal{T}_g|^2 = |\mathcal{T}_g|^2. \tag{181}$$

For a label-independent source basis $V_g = V$, the action transforms by conjugacy. The spin-one Gaussian witness implies

$$\boxed{VU_{1,0}V^{-1} \neq I} \tag{182}$$

for every global unitary basis V.

For arbitrary label-dependent measurable source frames, define the negative lexicographic half-sheet

$$\mathcal{F}_- := \{(J, \nu) : J < 0\} \cup \{(0, \nu) : \nu < 0\}. \tag{183}$$

Choose

$$V_g = \begin{cases} U_g, & g \in \mathcal{F}_-, \\ I, & g \notin \mathcal{F}_-. \end{cases} \tag{184}$$

This family is strongly measurable on each fixed-spin frequency row and satisfies

$$\boxed{U_g^V = I \qquad \forall g \in \Gamma.} \tag{185}$$

The classification is therefore precise. The source action is explicit and nontrivial in the source coordinates inherited from the physical kernel and remains nontrivial under every global source-basis change. As a bare measurable two-sheet cocycle it admits a label-dependent trivialization. The positive-measure isometry obstruction is stated for the normalized physical kernel in the kernel-inherited source coordinates; scalar rephasings preserve its modulus.

16 Summary of results

The results can be summarized in three layers.

Kernel-derived joint representation. The normalized physical Clebsch–Gordan kernel determines the source character U_g, the spectral translation $\Theta_{J,\nu}$ and the joint source/output Weyl involution. Its phase-bearing fixed subspace is

$$\boxed{\mathcal{H}_{\text{joint}}^{\text{fix}} = \{F \in \mathcal{H}_{\text{joint}} : \mathsf{W}F = F\}.} \tag{186}$$

Restriction to the fundamental output domain gives

$$\boxed{\mathcal{H}_{\text{joint}}^{\text{fix}} \simeq_u \mathcal{H}_{\text{red}}.} \tag{187}$$

This removes the twofold signed output-Weyl duplication and retains the entire source spectral space.

Positive-measure obstruction to fixed-source isometry. The spin-one transition obeys

$$\mathcal{T}_3(0,0,0) = 1 - 4\log 2 \tag{188}$$

and has Gram factor greater than one on an open positive-measure subset of the joint external/output spectral parameter space. Every fiberwise isometric realization requires unit Gram. Hence Eq. (168) holds.

Completion and source-frame classifications. Regular-locus distribution theory gives $D_c = D_{\text{can}} + c\delta_0$ and $\mathcal{T}_c = \mathcal{T}_{\text{can}} + c$. The homogeneous/Fourier-normalized shadow extension selects $c = 0$. The source representation is nontrivial under scalar rephasing and global source-basis changes up to the corresponding equivalence, and an explicit measurable label-dependent source-frame gauge trivializes the bare two-sheet cocycle.

17 Interpretation

The central representation-theoretic content is the kernel-derived source action. In the source coordinates inherited from the physical kernel, Weyl/shadow reflection acts by

$$(J, \nu, \xi) \longmapsto (-J, -\nu, \Theta_{J,\nu}\xi), \tag{189}$$

with the channel phase carried by the fixed-subspace law. The Weyl-fixed joint L^2 space records this source/output representation geometry. The fundamental-domain unitary is its Hilbert-space consequence.

The source-frame classification identifies the scope of the source-aware terminology. The kernel-inherited source coordinates carry the explicit character in Eq. (179). Scalar row phases leave the Gram factor unchanged. Global source-basis changes preserve source-action nontriviality by conjugacy. A label-dependent measurable source frame can trivialize the bare two-sheet cocycle. Thus the canonical kernel representation and the abstract measurable cocycle class are distinct mathematical structures.

The negative result has a separate metric basis. The normalized transition produces the Gram factor

$$|\mathcal{T}|^2. \tag{190}$$

Fiberwise isometry requires this factor to equal one. The full three-frequency theorem finds an open positive-measure region where it is strictly larger than one. Scalar row phases preserve the modulus, and the homogeneous/Fourier-normalized distributional completion fixes the point-supported extension freedom. The isometry obstruction therefore belongs to the normalized physical kernel at the stated completion convention.

The contact classification makes the normalization boundary explicit. Regular-locus agreement permits the affine family in Eq. (170); the homogeneous shadow law selects the unique physical representative. This separation exposes precisely which data determine the modulus.

Finally, the term *Weyl-reduced* has a narrow meaning: one representative of each signed output Weyl pair is retained, and the source spectral space remains. At the bounded operator-family level the reflected datum is

$$\mathcal{C}_{-J,-\nu}[U^{\mathrm{src}}_{J,\nu}\psi], \tag{191}$$

and the fixed-source opposite row is

$$\mathcal{C}_{-J,-\nu}[\psi]. \tag{192}$$

The physical angular transform is assembled from the literal Clebsch–Gordan kernel as almost-everywhere spectral rows in a completed direct-integral carrier. Paper 6 develops the associated row realization and range geometry.

18 Relation to Papers 4 and 6

Paper 4 hands forward the qualified physical radiative Bose pair

$$\Phi^{(2)} \in \mathcal{H}^{(2)}_{\mathrm{phys}}. \tag{193}$$

On the dense kernel core, Paper 5 resolves this same two-particle input at fixed external labels by

$$\boxed{\Psi \in \mathcal{D}_{\mathrm{pair}} \xmapsto{\iota_{a,b}} \psi_{a,b} \in \mathcal{H}_{\mathrm{src}}.} \tag{194}$$

The representation problem then pairs

$$\boxed{(J, \nu, \psi_{a,b}) \longleftrightarrow (-J, -\nu, U^{\mathrm{src}}_{J,\nu}\psi_{a,b}),} \tag{195}$$

with Hilbert-space reduction

$$\boxed{\mathcal{H}^{\mathrm{fix}}_{\mathrm{joint}} \simeq_u \mathcal{H}_{\mathrm{red}}.} \tag{196}$$

Paper 6 fixes the source vector $\psi_{a,b} = \iota_{a,b}\Psi$ and constructs the physical angular coefficient from the literal kernel. On the dense core the resulting spectral data are assembled as

$$\boxed{\Psi \longmapsto \Big[(J, \nu) \longmapsto C^{a,b,\mathrm{core}}_{J,\nu}[\Psi]\Big]_{L^2},} \tag{197}$$

where the brackets denote the completed direct-integral equivalence class and the displayed rows are representative data defined almost everywhere in the continuous spectral coordinate.

Paper 5 supplies the bounded source-aware operator-family reconstruction

$$\boxed{\mathcal{C}_{-J,-\nu}[U^{\mathrm{src}}_{J,\nu}\psi_{a,b}] \qquad \text{and} \qquad \mathcal{C}_{-J,-\nu}[\psi_{a,b}],} \tag{198}$$

for the Weyl-paired and fixed-source operator rows. Paper 6 studies the additional analytic realization problem connecting physical a.e. Clebsch–Gordan rows with completed range geometry.

The completed angular shadow $\mathcal{S}_{-\nu,J}$ in Eq. (75) acts on the output angular Hilbert fiber and is completed in the output-shadow variable $w \in \mathbb{C}_{\mathrm{sh}}$ of Eq. (133). Downstream source-incidence completion belongs to a separate coordinate problem, and downstream coordinate label reflection remains a separate map. Paper 5 retains $\mathcal{S}_{-\nu,J}$ as the completed output-fiber shadow operator throughout.

The sequence from Papers 4–6 is therefore

$$\boxed{\begin{array}{c} \Phi^{(2)} \in \mathcal{H}^{(2)}_{\mathrm{phys}}, \qquad \mathcal{D}_{\mathrm{pair}} \subset \mathcal{H}^{(2)}_{\mathrm{phys}}, \qquad \Psi \in \mathcal{D}_{\mathrm{pair}} \xrightarrow{\iota_{a,b}} \psi_{a,b} \in \mathcal{H}_{\mathrm{src}} \\ \downarrow \text{ Paper 5: } (J, \nu, \psi) \leftrightarrow (-J, -\nu, U^{\mathrm{src}}_{J,\nu}\psi) \\ \mathcal{H}^{\mathrm{fix}}_{\mathrm{joint}} \simeq_u \mathcal{H}_{\mathrm{red}} \\ \downarrow \text{ Paper 6: fixed } \psi_{a,b} \\ \Big[(J, \nu) \mapsto C^{a,b,\mathrm{core}}_{J,\nu}[\Psi]\Big]_{L^2}. \end{array}} \tag{199}$$

19 Conclusion

The normalized radiative two-particle Clebsch–Gordan kernel determines an explicit source-space representation: the input-chord character becomes the spectral translation

$$(\omega, k) \longmapsto \left(\omega + \frac{\nu}{\pi}, k + 2J\right),$$

and the combined source/output Weyl involution defines a Weyl-fixed joint spectral subspace unitarily equivalent to one fundamental output domain. Its $\sqrt{2}$ normalization is the norm identity of the Weyl-paired data (J, ν, ψ) and $(-J, -\nu, U^{\mathrm{src}}_{J,\nu}\psi)$. The reduction removes signed output-Weyl duplication and retains the full source Hilbert space.

The contact-normalized transition obeys a stronger metric theorem. Its equal-helicity spin-one value is $1-4\log 2$, an independent reconstruction reproduces that value from the normalized kernel, and joint continuity extends $|\mathcal{T}|^2 > 1$ to an open positive-measure subset of the joint external/output spectral parameter space. Fiberwise isometry requires unit Gram, yielding the almost-everywhere fixed-source isometry obstruction. Regular-locus contact freedom in the output-shadow plane forms the affine family $D_{\mathrm{can}} + c\delta_0$; the homogeneous/Fourier-normalized shadow extension selects $c = 0$. The source representation remains nontrivial under global source-basis changes and admits a label-dependent measurable trivialization as a bare two-sheet cocycle. Paper 5 therefore hands Paper 6 the Weyl-fixed joint representation, the bounded source-aware operator-family reconstruction, the kernel-derived source unitary and shadow data, and the fixed-source distinction displayed in Eq. (198), together with the isometry, completion and source-frame classifications. Paper 6 then supplies the separate direct-integral completion and range analysis of the literal angular Clebsch–Gordan transform.

A Space and notation ledger

Notation	Role	Formal counterpart
$\mathcal{H}^{(2)}_{\mathrm{phys}}$	physical radiative Bose pair	`PhysicalRadiativeTwoParticle`
$\mathcal{H}_{\mathrm{src}}$	fixed-external-label source Hilbert space	`PhysicalCGFixedInputSourceHilbert`
$\mathcal{H}^{\mathrm{ang}}_{J}$	angular output fiber	integral-spin celestial angular Hilbert fiber
$\mathcal{H}^{\mathrm{red}}_{\mathrm{ang}}$	angular Weyl-reduced coefficient space	`PhysicalCompositeMultiplicityFreeSpectrum`
$\mathcal{H}_{\mathrm{joint}}$	scalar joint output/source spectral carrier	`PhysicalCGSourceTwistedJointSpectralHilbert`
$\mathcal{H}_{\mathrm{red}}$	scalar L^2 space on a Weyl fundamental domain with the full source space retained	`PhysicalCGSourceTwistedMultiplicityFreeHilbert`
$\mathscr{W}$	joint Weyl/source shear	`physicalCGSourceTwistedJointSpectralShearEquiv`
$\gamma_{a,b}(J,\nu)$	all-label phase	`physicalCGStarTriangleAllLabelShadowCoefficient`
$U^{\mathrm{src}}_{J,\nu}$	input-chord unitary	`physicalCGInputChordTransitionUnitary`
$\mathcal{T}^{a,b}_{J,\nu}$	contact-normalized Clebsch–Gordan transition coefficient	`physicalCGLiteralFixedInputActualNormalizedTransitionCoefficient`

B Weyl-fixed analysis and synthesis

For completeness, the two central maps are collected here:

$$\mathcal{A}_{a,b}F = \sqrt{2}\,RF, \tag{200}$$

and

$$\mathcal{S}_{a,b}f = \frac{1}{\sqrt{2}}\left(Ef + \mathsf{W}_{a,b}Ef\right). \tag{201}$$

They satisfy

$$\mathcal{A}_{a,b}\mathcal{S}_{a,b} = \mathbf{1}, \qquad \mathcal{S}_{a,b}\mathcal{A}_{a,b} = \mathbf{1}, \tag{202}$$

and

$$\langle \mathcal{A}_{a,b}F, \mathcal{A}_{a,b}G\rangle = \langle F, G\rangle. \tag{203}$$

C Historical retained-sheet conditions

The historical five-condition proposition can be represented schematically as

$$C_{\mathrm{hist}} = C_{\mathrm{R}} \wedge C_{\mathrm{S}} \wedge C_{\mathrm{P}} \wedge C_{\mathrm{sep}} \wedge C_{\mathrm{G}}. \tag{204}$$

The exact defect relation

$$2D_{\mathrm{P}} = D_{\mathrm{R}} + D_{\mathrm{S}} \tag{205}$$

shows why the individual retained-Parseval question is kept separate.

The theorem

$$\neg\,\text{normalized transition unit Gram} \tag{206}$$

implies

$$\neg C_{\text{hist}} \tag{207}$$

and leaves the other fields at their independently established status.

D Explicit spin-one spherical-weighted radial reduction

This appendix exposes the elementary one-dimensional calculation behind Eq. (151). Set

$$x_1 = \frac{3-\sqrt{5}}{2}, \qquad x_2 = 3-\sqrt{5}. \tag{208}$$

The three Jacobian-weighted radial densities are

$$f_{\mathrm{I}}(x) = -\frac{1}{8(1-x)} + \frac{1}{2(1-x)^2} + \frac{3}{4(2-x)} - \frac{1}{4(2-x)^2}, \tag{209}$$

$$f_{\mathrm{M}}(x) = -\frac{1}{2} - \frac{1}{8(1-x)} + \frac{3}{4(2-x)} - \frac{1}{4(2-x)^2}, \tag{210}$$

$$f_{\mathrm{O}}(x) = -\frac{1}{2} + \frac{1}{8x} + \frac{5}{8(2-x)} - \frac{1}{4(2-x)^2}. \tag{211}$$

Elementary primitives are

$$F_{\mathrm{I}}(x) = \frac{1}{8}\log(1-x) + \frac{1}{2(1-x)} - \frac{3}{4}\log(2-x) - \frac{1}{4(2-x)}, \tag{212}$$

$$F_{\mathrm{M}}(x) = -\frac{x}{2} + \frac{1}{8}\log(1-x) - \frac{3}{4}\log(2-x) - \frac{1}{4(2-x)}, \tag{213}$$

$$F_{\mathrm{O}}(x) = -\frac{x}{2} + \frac{1}{8}\log x - \frac{5}{8}\log(2-x) - \frac{1}{4(2-x)}. \tag{214}$$

A direct differentiation verifies $F_{\mathrm{I}}' = f_{\mathrm{I}}$, $F_{\mathrm{M}}' = f_{\mathrm{M}}$ and $F_{\mathrm{O}}' = f_{\mathrm{O}}$ on the relevant intervals. Therefore

$$M = \big[F_{\mathrm{I}}(x)\big]_0^{x_1} + \big[F_{\mathrm{M}}(x)\big]_{x_1}^{x_2} + \big[F_{\mathrm{O}}(x)\big]_{x_2}^{1}. \tag{215}$$

The algebra reduces using

$$\frac{x_1}{2} + \frac{1}{2(1-x_1)} = 1 \tag{216}$$

and

$$\log(1-x_2) - \log x_2 - \log(2-x_2) = \log\left(\frac{1-x_2}{x_2(2-x_2)}\right) \tag{217}$$

$$= \log\frac{1}{4} = -2\log 2. \tag{218}$$

Substitution into Eq. (215) yields

$$\boxed{M = -\frac{1}{8} + \frac{1}{2}\log 2.} \tag{219}$$

The normalized shadow/contact factor is -8, hence

$$\boxed{\mathcal{T}_{1,0} = -8M = 1 - 4\log 2.} \tag{220}$$

This is the numerical witness used in the positive-measure Gram classification. [1]

D.1 Independent original-kernel normalization audit

The independent audit begins from the original CI194 kernel and reconstructs the normalization without invoking the specialized spin-one endpoint theorems. Its exact ledger is

$$N_{\mathrm{CG}} = \frac{1}{8\pi^2}, \qquad \rho_{\mathrm{Pl}}(1,0) = 8, \qquad \sqrt{\rho_{\mathrm{Pl}}(1,0)} = 2\sqrt{2}, \tag{221}$$

$$P_{\mathrm{ang}} = 2\pi, \qquad S_{1,0} = -\frac{1}{\pi}, \qquad A_{\mathrm{contact}}(0) = \frac{N_{\mathrm{CG}}}{2}. \tag{222}$$

The normalized round-sphere constant has amplitude $(4\pi)^{-1/2}$ and its flat stereographic image has unit norm. The corresponding unnormalized outer factor has norm $\sqrt{4\pi}$.

The contact-normalized weighted prefactor is therefore

$$\frac{-2N_{\mathrm{CG}}/\pi}{N_{\mathrm{CG}}/2} = -\frac{4}{\pi}. \tag{223}$$

The direct unweighted contour kernel follows algebraically from the independently specialized CI194 exponents. Fresh partial fractions give the angular means

$$\frac{7}{8}, \qquad -\frac{1}{8}, \qquad 0, \tag{224}$$

and hence the unweighted endpoint -1.

For the spherical-weighted kernel, direct contour factorization yields the moving-pole relation

$$r(x)^2 = \frac{x(2-x)}{1-x} \tag{225}$$

and exact chamber identities

$$\frac{R'(x)}{2} A_{\mathrm{sph}}(x, r(x)) = \begin{cases} f_{\mathrm{I}}(x), & 0 < x < x_1, \\ f_{\mathrm{M}}(x), & x_1 < x < x_2, \\ f_{\mathrm{O}}(x), & x_2 < x < 1. \end{cases} \tag{226}$$

The weighted kernel itself has a second-order contact puncture, and the contour-selected finite-part plane functional is

$$I^{\mathrm{FP}}_{\mathrm{sph}} = 2\pi M. \tag{227}$$

Consequently

$$\boxed{\mathcal{T}^{\mathrm{ind}}_{1,0} = \left(-\frac{4}{\pi}\right)\left[2\pi\left(-\frac{1}{8} + \frac{1}{2}\log 2\right)\right] = 1 - 4\log 2.} \tag{228}$$

[1] *Formal counterpart:* `Tier30.Research.CelestialInformationGeometry.CI206.CanonicalPhysicalCGLiteralPairCoreWeightedSpinOneZeroFrequency`.

The contact-cancelled relative integrand is globally integrable. The structural binding proves almost-everywhere equality

$$R^{\mathrm{ind}}_{\mathrm{rel}}(u) = R^{\mathrm{actual}}_{\mathrm{rel}}(u) \qquad \text{a.e.,} \tag{229}$$

then equality of the unevaluated integrals, followed by the closed-form evaluation. The unweighted finite-part integral is likewise identified with the actual unweighted completed coefficient before the final phase ratio is taken. These two bindings give Eq. (158).[2]

E Uniform majorants for the three-frequency continuity step

The full spectral result varies the two incoming Mellin frequencies and the output principal-series frequency. Write

$$p = (\nu_1, \nu_2, \nu) \in [-1,1]^3.$$

The pointwise kernel factors are jointly continuous in p away from the three null punctures $u = 0, -1, -2$. Compactness of the parameter cube supplies uniform bounds for the resolved Gamma ratios, inverse Gamma factors and contact denominator. A compact C^1 estimate supplies one Lipschitz constant for the normalized contact subtraction over the same cube.

For $q, r > 0$, write

$$B_{q,r}(u) := \mathbf{1}_{\{|u|<r\}} |u|^{-q}, \tag{230}$$

and let

$$C_R(u) := \mathbf{1}_{\{|u|\le R\}}. \tag{231}$$

The common puncture/exterior envelope is

$$\boxed{E(u) = 64 \left[B_{1,1/4}(1+u) + B_{1,1/4}(2+u) + \mathbf{1}_{\{|u|>4\}} |u|^{-4} + C_4(u) \right].} \tag{232}$$

Each term is integrable on $\mathbb{C} \simeq \mathbb{R}^2$; for example,

$$\int_{|u|<r} |u|^{-1}\, d^2u = 2\pi r, \qquad \int_{|u|>R} |u|^{-4}\, d^2u = \frac{\pi}{R^2}. \tag{233}$$

Let G be a uniform bound for the relevant inverse Gamma factors on the cube, and let L_{fp} and L_{rel} be uniform contact Lipschitz constants for the finite-part and relative-correction amplitudes. The same integrable shapes used in the one-frequency precursor provide cube-uniform majorants,

$$M_{\mathrm{fp}}(u) := G L_{\mathrm{fp}} B_{1,1}(u) + 2GE(u) + 4GC_2(u), \tag{234}$$

$$M_{\mathrm{rel}}(u) := 8 G L_{\mathrm{rel}} B_{1,1/2}(u) + 6GE(u). \tag{235}$$

For every $p \in [-1,1]^3$ and almost every u,

$$|I_{\mathrm{fp}}(p,u)| \le M_{\mathrm{fp}}(u), \qquad |I_{\mathrm{rel}}(p,u)| \le M_{\mathrm{rel}}(u), \tag{236}$$

with

$$M_{\mathrm{fp}}, M_{\mathrm{rel}} \in L^1(\mathbb{C}, d^2u). \tag{237}$$

[2] *Formal counterpart:* `Tier30.Research.CelestialInformationGeometry.CI212.CanonicalPaper5IndependentSpinOneStructuralActualBinding`.

These bounds, together with joint pointwise continuity away from the null punctures, give the dominated-convergence argument used in Eq. (162). The compact parameter cube also supplies the nonvanishing contact denominator required by the normalized quotient.[3]

F Formal verification map

The main text uses invariant mathematical language. Repository names below identify the exact formal counterparts.

The original complete Paper 5 authority is pinned to repository commit

`47a8041ce05abe2e6bf2f5112e4d9b1da4bca811`.

The later carrier-semantic clarification used in Sections 2, 7 and 17 is recorded by the Paper 5/Paper 6 audit introduced at commit

`c616c04f395b3b34d1f80ec5ef6de327855f82cc`.

The independent spin-one normalization audit closes at

`a34587f2f3d679e61aa0c1748f7ec792fe27e610`,

and the structural kernel-to-radial and actual-transition binding closes at

`fdb1cd91c7659a142eaf128954fc6724460421ae`.

[3] *Formal counterpart:* `Tier30.Research.CelestialInformationGeometry.CI212.CanonicalPaper5FullSpectralGramRigidity`.

Core mathematical result	Formal authority
Source-aware shadow normalization	`CanonicalPhysicalCGSourceTwistedKernelShadowCompletion`
Measure-preserving joint source/output shear	`CanonicalPhysicalCGSourceTwistedJointSpectralShear`
Phase-bearing fixed sector	`CanonicalPhysicalCGSourceTwistedJointPhasedSpectralSymmetrization`
Forced-$\sqrt{2}$ analysis and explicit synthesis	`CanonicalPhysicalCGSourceTwistedPhasedMultiplicityFreeEquivalence`
Parseval and both inverse identities	`paper5SourceAwareCelestialCGCompletion`
External coefficient measurability	`CanonicalPhysicalCGLiteralStarTriangleAllLabelCoefficient`
Global external phase unitary	`paper5ExternalChannelPhaseAssembly`
Source-aware operator-family descent	`CanonicalPhysicalCGSourceTwistedOperatorFamilyDescent`
Source-aware operator-family equivalence	`CanonicalPhysicalCGSourceTwistedOperatorFamilyEquivalence`
Raw/Plancherel-normalized coefficient distinction	`CanonicalPhysicalRadiativeCelestialCompositeSurface`
Direct signed-row deletion nontrivial kernel	`CanonicalPhysicalCGDirectRestrictionParsevalNoGo`
Actual-CI194 comparison/deletion incompatibility	`CanonicalPhysicalCGLiteralDirectComparisonBridgeNoGo`
Affine two-chord distributional/non-L^2 behavior	`CanonicalPhysicalCGLiteralStarTriangleAllSpinAffineTwoChordNotL2`
Phase-only/source-twisted conjunction negation	`CanonicalPaper5OriginalSameSourceContractAudit`
Exact contact-defect decomposition and cofinal exhaustion	`CanonicalPhysicalCGLiteralStarTriangleAllSpinJointContactDefect;CanonicalPhysicalCGLiteralStarTriangleAllSpinJointContactCofinalExhaustion`
Synchronized pair-core Gram diagonal	`CanonicalPhysicalCGLiteralStarTriangleAllSpinPhasedPairCorePairedGramDiagonalLimit`
Pair-core completed Gram factorization	`CanonicalPhysicalCGLiteralPairCoreCompletedGramFactorization`
Contact normalization	`CanonicalPhysicalCGLiteralPairCoreUniversalScalarContactNormalization`
Spin-one unweighted residue	`CanonicalPhysicalCGLiteralPairCoreSpinOneZeroFrequencyResidue`
Spin-one spherical-weighted residue $1 - 4\log 2$	`CanonicalPhysicalCGLiteralPairCoreWeightedSpinOneZeroFrequency`
Actual Bochner/contact binding	`CanonicalPhysicalCGLiteralPairCoreWeightedSpinOneZeroFrequencyBinding`
Frequency continuity and positive-measure Gram inequality	`CanonicalPhysicalCGLiteralPairCoreFrequencyContinuity`
Actual transition Gram inequality	`not_physicalCGLiteralFixedInputActualTransitionGramPreservation`
Five-condition proposition negation / Lean realization-type emptiness	`not_paper5LiteralFixedInputPublicationContract;not_nonempty_paper5LiteralFixedInputPublicationContract`

Independent verification and semantic result	Formal authority
Independent CI194 normalization ledger and direct endpoint	`CanonicalPaper5IndependentSpinOneKernelReconstruction;CanonicalPaper5IndependentSpinOneWeightedEndpoint`
Independent contour residues and kernel-to-density identities	`CanonicalPaper5IndependentSpinOneKernelRadialBinding`
Independent finite-part plane functional and hardened scalar	`CanonicalPaper5IndependentSpinOnePlaneIntegral`
Unevaluated relative-integrand equality and structural actual binding	`CanonicalPaper5IndependentSpinOneStructuralActualBinding;Paper5IndependentSpinOneStructuralAuthority`
Independent verification script	`scripts/verify_paper5_independent_spin_one_hardening.py`
Complete Paper 5 authority	`paper5CompleteCelestialCGAuthority`
P5/P6 carrier semantic ledger	`paper5Paper6KernelSemanticLedger`
Scalar P5 carrier classification	`paper5SourceAwareCarrierSemanticDecision_exact`
Actual angular-output/P5 carrier separation	`paper5ActualCI194CarrierSeparation`
Three-frequency continuity, positive external/output measure, and isometry obstruction	`CanonicalPaper5FullSpectralGramRigidity`
Cutoff independence, Dirac extension classification, and homogeneous completion rigidity	`CanonicalPaper5CompletionRigidity;Paper5CompletionRigidityDecision`
Source representation and source-frame trivialization classification	`CanonicalPaper5SourceGaugeClassification`
Final Paper 5 formal authority	`Paper5FinalRefereeHardeningAuthority`
Final verification script	`scripts/verify_paper5_final_referee_hardening.py`

G Formal development and verification boundary

The publication-facing body uses standard mathematical-physics terminology. Repository declarations appear here only as provenance.

The Weyl-fixed analysis/synthesis theorem is collected by

`Tier30.Research.CelestialInformationGeometry.CI212.CanonicalPaper5SourceAwareCelestialCGCompletion`.

The original complete Paper 5 formal package is

`Tier30.Research.CelestialInformationGeometry.CI212.CanonicalPaper5CompleteCelestialCGAuthority`.

The independent spin-one derivation reconstructs the equal-helicity $(J, \nu) = (1, 0)$ normalization from the physical Clebsch–Gordan kernel along a dependency-independent path. It recomputes the kernel exponents, Plancherel density, sphere/plane normalization, shadow factor, contact amplitude, contour residues, moving-pole geometry and kernel-to-density identities, then identifies the unevaluated relative integrand with the completed transition before closed-form evaluation. Its structural endpoint is `Paper5IndependentSpinOneStructuralAuthority`.

The final Paper 5 formal package is

`Tier30.Research.CelestialInformationGeometry.CI212.CanonicalPaper5FinalRefereeHardeningAuthority`.

It records joint continuity in both incoming Mellin frequencies and the output frequency, the open positive-measure external/output Gram defect, the theorem that fiberwise isometry forces unit Gram, the almost-everywhere isometry obstruction, normalized-cutoff independence, the unique spin-one Dirac extension coordinate, the exact counterterm law $\mathcal{T}_c = \mathcal{T}_{\text{can}} + c$, homogeneous-extension rigidity $c = 0$, the strongly continuous additive source representation, global-basis nontriviality, measurable source-frame trivialization and the publication semantics of the Weyl reduction.

The semantic crosswalk

`Tier30.Research.CelestialInformationGeometry.CI216.CanonicalPaper5Paper6KernelSemanticAudit`

records the distinct roles of the physical radiative pair space, source Hilbert space, angular coefficient space and Weyl-fixed scalar joint space. The body therefore makes no range-completion claim for the angular transform.

The relevant repository milestones are:

Purpose	Commit
Original complete Paper 5 package	`47a8041ce05abe2e6bf2f5112e4d9b1da4bca811`
Carrier-semantic clarification	`c616c04f395b3b34d1f80ec5ef6de327855f82cc`
Independent spin-one normalization	`a34587f2f3d679e61aa0c1748f7ec792fe27e610`
Structural kernel-to-radial identification	`fdb1cd91c7659a142eaf128954fc6724460421ae`
Full spectral/completion/source-frame verification	`0baed265914b0e75848f2c2d0c80d6f8932269ad`

The checked toolchain is Lean 4 `v4.28.0-rc1` with mathlib revision

`5352afccd6866369be9de43f5b7ec47203555f44`.

The verification scripts inspect theorem surfaces, dependency boundaries, provenance, shortcut exclusions and proof hygiene. The numerical audit compares complex-plane polar quadrature with the independently derived chamber reduction before the exact target value is introduced.

The final verification module has no CI217-or-later dependency. The inspected Paper 5 verification modules contain no `sorry`, `admit`, newly declared `axiom`, or `native_decide`.

Machine verification establishes that the formalized mathematical conclusions follow from the encoded definitions, premises and transitive dependencies. Physical interpretation is supported separately by the model specification, analytic normalization, continuum assumptions and empirical evidence appropriate to each claim.

The manuscript closes at the global two-particle representation level. Coincident-point local CFT OPEs, interacting collinear OPEs, detector tomography, soft-current algebra, $w_{1+\infty}$ realization and faithful boundary reconstruction belong to later stages of the series.

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